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NON-UNIQUENESS AND LIMITING ASPECTS OF AUTOMATIC HISTORY-MATCHING

by

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(C)

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ABSTRACT

An intuitive trial-and-error approach to history-matching can be costly and time-consuming. Considerable efforts have therefore been made to automate history-matching procedures for implementation on high-speed computers.

Several such automatic history-matching methods have been reported, but, because their effectiveness and reliability remain to be demonstrated, they have not been well-received by practising reservoir engineers.

This study examines presently available automatic history-matching algorithms, discusses those that appear most viable, and explores the fundamental weaknesses of these methods. In addition, the existence of alternative approaches to solving the inverse problem for analogous, but less complex, situations is noted.

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NOMENCLATURE

ENGLISH

B = formation volume factor, m^3/sm^3

c_r = rock compressibility, Pa^{-1}

D = depth, measured vertically positive downwards, m

g = acceleration due to gravity, m/s^2

k = absolute permeability, m^2

k_r = relative permeability, fraction

P_{cgo} = gas-oil capillary pressure, Pa

P_{cow} = oil-water capillary pressure, Pa

p_f = phase pressure, Pa

q_f^* = phase fluid production(-)/injection(+) rate, sm^3/s

R_{so} = solubility of gas in liquid phase, sm^3/sm^3

R_{sw} = solubility of gas in water phase, sm^3/sm^3

S_f = phase saturation, fraction

t = time, s

V_b = bulk volume, m^3

x_{fa} = mole fraction of oil, water, or gas in aqueous phase

x_{fl} = mole fraction of oil, water, or gas in liquid phase

x_{fv} = mole fraction of oil, water, or gas in vapour phase

Greek

Δt = time step, $t^{n+1} - t^n$, s

$\Delta x, \Delta y, \Delta z$ = block dimensions in x,y,z directions, m

μ = viscosity, Pa·s

ρ = density, kg/m^3

$\overline{\rho}$ = molar density, kg-mole/m^3

Φ = potential, Pa

ϕ = porosity, fraction

Subscripts

a = aqueous phase

f = phase

g = gas

i,j,k = x,y,z direction nodal subscripts

l = liquid phase

o = oil

v = vapour

w = water

x,y,z = x,y,z directions

Superscripts

n = present time level index

n+1 = advanced time level index

CHAPTER ONE

INTRODUCTION

Increasingly sophisticated high-speed computing capabilities and attendant development of efficient numerical algorithms have made it possible to utilize complex mathematical models for analysis, development, and management of petroleum reservoirs. Such models can represent the simultaneous flow of oil, water, and gas, and the mutual interactions of these fluids, within the porous and permeable strata which comprise a reservoir.

A reservoir simulator is used, first and foremost, for predicting the volume of oil recoverable by various production schemes. A wide spectrum of different operating parameters can thus be tested; and provided that the geology of the reservoir is known in sufficient detail, and a good estimate of the porosity and permeability of the rock can be made, the simulation results can then be used to formulate an optimal development strategy for a particular field.

Rough estimates of the rock properties may be obtained from analysis of core samples taken from the reservoir. However, due to high costs of drilling on closely spaced centres - e.g., 15-20 m apart - the number of samples, is usually small; and for a large reservoir which is likely to be heterogeneous, such estimates are, at best, little more than educated guesses. Some refinement of the estimates is possible if other known data - i.e. the performance histories of the reservoir - are incorporated. Such so-called 'history-matching' (or 'inverse simulation') involves using a reservoir simulator to 'adjust', within

set bounds, future production/injection and pressure parameters to values consistent with previous reservoir behaviour. More specifically, it requires iterative computer simulations, with each run adjusting the primary input data (porosity, permeability, and relative permeability) until an acceptable 'match' has been obtained.

While the concept of history-matching is straightforward, the achievement of a good match is difficult, primarily because of the non-uniqueness of the solution (see, e.g., Chapter 2). In practice, any history-match therefore seeks to determine the 'best' set of input parameters within pre-determined bounds (which are usually based on estimates from core samples).

Although history-matching is frequently performed by reservoir engineers on an intuitive, trial-and-error basis, considerable efforts have been made to design 'automatic' history-matching algorithms, i.e., to program, in some fashion, a computer to determine an optimal set of reservoir parameters. But despite these attempts to simplify (and speed up) the entire history-matching procedure, extant techniques appear not to have been well-received by practising reservoir engineers - possibly due to perceived shortcomings of such techniques. This study therefore examines the various, presently available, automatic history-matching algorithms, and evaluates their relative merits and weaknesses as viable alternatives to non-automatic techniques.

CHAPTER TWO

REVIEW OF THE LITERATURE

First attempts to estimate the geological properties of a petroleum reservoir for use in reservoir simulation studies were reported by Kruger (1961). He emphasized the importance of obtaining agreement between calculated and observed pressure histories, and suggested that reservoir properties be adjusted by small, successive perturbations, with a simulator run undertaken after each iteration in order to determine systematically an 'acceptable' match.

Nelson (1962) provided a short analysis of the paper by Kruger (1961), and in particular, discussed conditions necessary to insure a unique and realistic determination of the permeability distribution.

Efforts to automate this procedure, and thus exploit the speed and efficiency of computers, were initiated by Jacquard (1964), and were predicated on the mathematics of an electrical resistance - capacitance network (known as an 'electric analyzer'). After demonstrating the close similarity between such a network and a reservoir model, Jacquard developed a method for interpreting pressure history data in terms of geological properties throughout a reservoir. The paper provides sample calculations of permeabilities for a simple, single-well reservoir with radial-circular heterogeneity, and indicates a pronounced sensitivity of results to minor variations in reservoir pressure.

Jacquard & Jain (1965) extended this technique, with the aid of least-squares methods, to the case of a two-dimensional reservoir. For this purpose, the reservoir was divided into zones, each of which was

assumed to have a constant permeability. (Similar zonation was subsequently adopted by other authors of automatic history-matching algorithms in order to reduce the number of unknown parameters.) The study used a reservoir simulator to generate an 'observed' pressure history, and then employed these data to compute reservoir parameters. It was found that, while retaining the original zonation led to excellent reconstitution of the reservoir, alternative zonations yielded poor estimates of reservoir properties.

Dupuy (1968) pursued the work of Jacquard & Jain (1965), and found instabilities reflected in the fact that calculated permeabilities frequently oscillated from one iteration to the next. He also noted that certain permeabilities were unrealistically high, and suggested the solution technique employed be coupled with imposition of some limiting values.

Jahns (1966), following the recommendations of Kruger (1961), Jacquard (1964), and Jacquard & Jain (1965), proposed the use of nonlinear regression analysis as means for selecting adjustment factors for reservoir parameters. An adaptation of the method of steepest descent, the technique was based on an analysis of effects that small, successive perturbations in each zone exert on performance histories predicted by a simulator. While the method was designed to handle single-phase flow, Jahns pointed out that it would also be suitable for multiphase flow if saturation changes were relatively small. Jahns was, incidentally, the first to consider the adjustment of porosity as well as permeability, and further recognized the non-uniqueness of his geological estimates. The paper applied the algorithm to two actual

reservoirs; and, in both cases, the average computed properties compared favorably with the, albeit sparse, available data. It should be noted, however, that there was a tendency to predict rather extreme values - according to Jahns, due to the short-term performance history used in his calculations.

Nelson (1968) approached the problem of determining reservoir permeability by incorporating an energy dissipation analysis. He examined the equation for single-phase flow, and reduced the resulting first-order partial differential equation in the unknown permeability to a system of characteristic equations; and this was then used to obtain a differential expression for permeability as a function of the known pressure distribution. For steady flow, for which the expression could be integrated, direct calculation of the permeability distribution along successive streamlines was possible. However, although this method produced excellent results for several single-phase test reservoirs, it is now of little practical value: the need for additional estimation of porosity, modelling of three-dimensional systems - and, most important, modelling of multiphase flow (where relative permeabilities must also be estimated) - leads to mathematical equations that are too complex to be solved by this method.

The work of Coats et al. (1970) led to the use of random selection of reservoir properties in a set of computer simulations. Assuming linear relationships between matching errors and reservoir properties, and calculating the coefficients for this relation by least-squares methods, these investigators employed linear programming to minimize the error and thus estimate both the reservoir porosity and permeability.

In several test cases, including a study of a two-phase reservoir, this procedure led to acceptable results. But although the system is basically nonlinear (Jacquard, 1964; Dupuy, 1968), the error-parameter relationship can be fairly well approximated with a linear function, especially where reservoirs possess relatively high porosities and permeabilities (Slater & Durrer, 1971; Carter et al., 1974). The principal advantage of the method of Coats et al. appears to lie in its ability to guarantee a global minimum error, as opposed to the local minima obtained by use of nonlinear optimization techniques.

Basing their work on the same general principles as Jacquard (1964), Dupuy (1968), and Coats et al. (1970), Slater & Durrer (1971) used error-weighted gradients and a linear programming formulation to systematically reduce differences between observed and calculated performance histories. In these studies, interference relationships were determined from the effects on performance histories of successively altering reservoir parameters. The interference relationships were then weighted and used to adjust the reservoir parameters. Slater & Durrer also discussed the impracticality of achieving a perfect match, especially when the accuracy of performance history measurements is questionable, and suggested reasonable guidelines for assessing when an acceptable match has been obtained. Examples involving hypothetical, single-phase reservoirs indicated a strong tendency to prediction of extreme values, and studies of actual reservoirs yielded poor results. Slater & Durrer consequently concluded that "in true reservoirs, where there exist uncertainties in pressure

measurements and the choice of a simulation model, a matching process is much more complicated."

A similar approach was employed by Thomas et al. (1972), who based their method on the classical Gauss-Newton least-squares procedure and parameter constraints, and made provision for handling highly nonlinear cases. They worked with data provided for three sample history-matchings used by Jahns (1966) and Coats et al. (1970) in order to compare methods and results, and found that while their algorithm in most cases required fewer simulator runs, it yielded matches almost identical to previously determined ones.

Boberg et al. (1973), using a variation of the algorithm by Thomas et al. (1972), were able to improve the stability of the numerical solution, and reported reasonable results for a complex Middle East oilfield.

Veatch & Thomas (1971) developed a novel direct method for handling the history-matching problem by treating the finite difference analogues of the partial differential equations for multiphase flow as a system of linear equations in the unknown reservoir properties. The technique is applicable to multiphase, compressible flow in heterogeneous reservoirs, and capable of calculating reservoir parameters in a single computer run. In addition, it contains provisions for constructing 'best' estimates where no unique solutions can be computed. However, relatively complete performance histories over varying periods of time are required as input data; and in most practical cases, where such data are lacking, it becomes necessary to employ interpolation schemes. Results of studies on several hypothetical reservoirs presented by

Veatch & Thomas were promising, but the method has not received widespread attention, and little commentary on the paper exists.

Carter et al. (1974), primarily expanding the work of Jacquard (1964) and Jacquard & Jain (1965), focussed attention on the method by which differences between observed and calculated performance histories are used to adjust reservoir properties, and also introduced two modified, iterative linear programming procedures. As a result, a method was designed for handling history-matching in situations where more reservoir parameters must be determined than are observed (known as an 'underdetermined' problem). Several examples of single-phase reservoirs were presented, and these indicated that linear programming formulations are able to yield reasonable estimates of reservoir parameters. For the case of underdetermined performance matching problem, a modified linear programming technique also produced acceptable results.

Chavent et al. (1973) and Chen et al. (1974) approached automatic history-matching by making use of optimal control theory, and estimated reservoir properties by continuous functions rather than as discrete values. Case studies in both papers were limited to single-phase reservoirs, for which the necessary governing equations could be easily derived. The method compared favourably with standard, constant-zone gradient methods (such as those used by Slater & Durrer, 1971; Thomas et al., 1972; and Carter et al., 1974), and on average, reduced the required computer time.

Wasserman et al. (1975) applied the optimal control methods of Chavent et al. (1973) and Chen et al. (1974) in an attempt to treat multiphase flow problems. Multiplication of the porosity and

permeability terms by saturation-dependent terms (obtained from running a multiphase simulator) seemed to transform, in effect, a multiphase reservoir to a 'pseudo' single-phase reservoir. However, while application of this technique to field reservoirs yielded acceptable results, none of the three studies cited above made any attempt to investigate the non-unique aspects of the solution in the context of optimal control theory. Nor did they discuss the likelihood of computing only a local minimum matching error.

An initial study of history-matching in two-phase petroleum reservoirs was reported by van den Bosch & Seinfeld (1977), who investigated the feasibility of estimating two-phase reservoir properties, and considered a simple, hypothetical reservoir with radial-circular symmetry, incompressible flow, and a central producing well. Qualitative and quantitative results suggested that the ability to estimate reservoir parameters depends on the type of flow within the reservoir, and evidenced non-uniqueness in the solutions of the history-matching problem.

A technique for history-matching based on estimation by Bayesian statistical methods was introduced by Gavalas et al. (1976). Analysis of a hypothetical, one-dimensional, single-phase reservoir indicated that estimation by such techniques produces better results than history-matching algorithms using zonation procedures. However, the accuracy of Bayesian estimates depends on the validity of the prior statistics (of reservoir properties) employed; and in practical situations, where detailed geological data are lacking, the procedure appears to be no more reliable, and probably less so, than earlier automatic history-matching methods.

Shah et al. (1978) presented a general analysis of errors arising in history-matching, with particular emphasis on error variation between Bayesian and constant-zone estimation techniques. From consideration of a hypothetical, one-dimensional, single-phase reservoir, they concluded that zonation techniques are clearly preferable when the location of zone boundaries is indicated by geological data. In cases where only sparse data on reservoir properties are available, neither Bayesian nor zonation procedures can be employed with much confidence.

Watson et al. (1980) further extended the optimal control algorithm of Chavent et al. (1973) and Chen et al. (1974) for automatic history-matching in two-phase reservoirs. They developed an algorithm for estimating relative permeability, as well as porosity and absolute permeability. Studies on hypothetical reservoirs produced reasonable estimates of reservoir properties. But the equations required for two-phase reservoir history-matching in this manner are extremely complex; and while it is theoretically possible to extend the algorithm to three-phase flow, practical considerations would seem to negate the feasibility of such a study.

CHAPTER THREE

DEVELOPMENT OF A PETROLEUM RESERVOIR SIMULATOR

Since a petroleum reservoir simulator is used as an essential component of automatic history-matching algorithms as well as for generating data that test the reliability of such algorithms, it is necessary to develop a suitable simulator before considering extant automatic history-matching techniques. This section discusses the construction of a three-phase (oil, water, gas), three-dimensional, compressible flow reservoir simulator in which oil is assumed to be non-volatible (such a simulator is known as a 'black oil' simulator).

Formulation of the Mathematical Model

A mathematical model that simulates multiphase, multi-dimensional fluid flow in a petroleum reservoir is conveniently based on Darcy's equation of flow and on the law of conservation of mass (known as the 'equation of continuity') as applied to each phase. These governing relations have been established by Muskat (1949), Collins (1961), Scheidegger (1974), Thomas (1982), and others.

Proceeding from these bases, a reservoir simulator for the present study was developed by assuming that

(1) flow is laminar, viscous, and irrotational;

(2) the flow process is isothermal;

(3) a thermodynamic equilibrium exists among the three phases;

(4) relative permeability curves adequately depict multiphase flow;

(5) phase interchanges are restricted to gas solution and release from oil and water; and

(6) the reservoir consists of a continuous porous medium which is externally bounded by an impermeable surface (i.e., there is no fluid flow across the outer boundaries of the reservoir).

The resulting simulator allows for effects of gravity, capillarity, fluid viscosity, relative permeabilities, gas solubility, and reservoir heterogeneity.

Consider a three-dimensional reservoir of variable thickness in the x,y,z framework of a Cartesian coordinate system, and let this reservoir contain a distribution of line sources and sinks (i.e., production and injection wells, respectively) oriented parallel to the z-axis, whose strengths are given as functions of time and location.

The general molar balance equation for each component i flowing in the oil (liquid), water (aqueous), and gas (vapour) phases through a porous medium is given by

$$\nabla\cdot\left(x_{il}\frac{kk_{rl}}{\mu_l}\bar{\rho}_l\nabla\Phi_l + x_{ia}\frac{kk_{ra}}{\mu_a}\bar{\rho}_a\nabla\Phi_a + x_{iv}\frac{kk_{rv}}{\mu_v}\bar{\rho}_v\nabla\Phi_v\right) + \frac{q_i^*}{\Delta x\Delta y\Delta z}$$

$$= \frac{\partial}{\partial t}\left[\phi\left(x_{il}\bar{\rho}_l S_l + x_{ia}\bar{\rho}_a S_a + x_{iv}\bar{\rho}_v S_v\right)\right], \qquad (3.1)$$

where $\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$, and the potential gradient for any phase f is given by

$$\nabla\Phi_f = \nabla p_f - \rho_f g\nabla D \ .$$

Assuming that phase interchanges are restricted to the solution and release of gas from oil and water, the mole fractions x_{oa}, x_{ov}, x_{wl}, and x_{wv} are all zero. The non-zero mole fractions are given by

$$x_{ol} = \frac{\bar{\rho}_{o,sc}}{B_o \bar{\rho}_l} \quad ,$$

$$x_{gl} = 1 - x_{ol} \quad ,$$

$$x_{wa} = \frac{\bar{\rho}_{w,sc}}{B_w \bar{\rho}_a} \quad ,$$

$$x_{ga} = 1 - x_{wa} \quad ,$$

and

$$x_{gv} = 1 \quad .$$

Substituting these values into the appropriate forms of Equation (3.1) yields the following phase equations for the simultaneous flow of oil, water, and gas:

Oil phase

$$\nabla \cdot \left(\frac{k k_{ro}}{\mu_o B_o} \nabla \Phi_o \right) + \frac{q_o^*}{\Delta x \Delta y \Delta z} = \frac{\partial}{\partial t} \left(\phi \frac{S_o}{B_o} \right) \quad , \qquad (3.2a)$$

Water phase

$$\nabla \cdot \left(\frac{k k_{rw}}{\mu_w B_w} \nabla \Phi_w \right) + \frac{q_w^*}{\Delta x \Delta y \Delta z} = \frac{\partial}{\partial t} \left(\phi \frac{S_w}{B_w} \right) \quad , \qquad (3.2b)$$

Gas phase

$$\nabla\cdot\left(R_{so}\frac{kk_{ro}}{\mu_o B_o}\nabla\Phi_o + R_{sw}\frac{kk_{rw}}{\mu_w B_w}\nabla\Phi_w + \frac{kk_{rg}}{\mu_g B_g}\nabla\Phi_g\right) + \frac{R_{so}\,q_o^* + R_{sw}q_w^* + q_g^*}{\Delta x\Delta y\Delta z}$$

$$= \frac{\partial}{\partial t}\left[\phi\left(R_{so}\frac{S_o}{B_o} + R_{sw}\frac{S_w}{B_w} + \frac{S_g}{B_g}\right)\right] . \qquad (3.2c)$$

If the porous medium of the reservoir is assumed to be fully saturated by the three phases, and preferentially water wet at the oil-water interface, the phase saturations are related by

$$S_o + S_w + S_g = 1 , \qquad (3.3)$$

and the capillary pressures by

$$P_{cow}(S_w) = p_o - p_w , \qquad (3.4)$$

$$P_{cgo}(S_w, S_g) = p_g - p_o . \qquad (3.5)$$

The unknowns in the model are the pressure and saturations. Moreover, if any one of the three other unknowns (q_o^*, q_w^*, q_g^*) in a block containing a source or sink is specified, the other two can be determined by making use of the relative mobility relationships

$$\frac{q_o^*}{q_w^*} = \frac{k_{ro}}{\mu_o B_o}\frac{\mu_w B_w}{k_{rw}} \qquad (3.6a)$$

and

$$\frac{q_o^*}{q_g^*} = \frac{k_{ro}}{\mu_o B_o} \frac{\mu_g B_g}{k_{rg}} . \qquad (3.6b)$$

Thus, a model which describes the flow mechanics of a petroleum reservoir is therefore provided by using the coupled Equations (3.2) in combination with Equations (3.3) - (3.6) and appropriate initial, boundary, and source-sink conditions. (For further discussion on development of this basic model, see, for example, Peaceman, 1969; Thomas, 1982; or Bird et al., 1960.)

Method of Solution

Due largely to the nonlinear nature of the system of partial differential equations, no analytical solution of Equations (3.2) is possible (Jacquard, 1964; Dupuy, 1968). Consequently, the technique of finite differences is frequently employed to obtain a numerical solution. (Finite differences have been extensively used in the petroleum industry for solving problems in porous media flow, and have yielded results that stand in good accord with experimental data.)

To utilize finite differences in the present case, the petroleum reservoir is defined by a block-centred grid that discretizes space variables, and the time continuum is subdivided into increments. Areal and cross-sectional sketches of a typical grid representation are shown in Figures 1 and 2. It is not necessary for the reservoir partitions to be equally spaced, and the blocks need not even be rectangular: depending upon the geological structure of the reservoir, circular, triangular, or curvilinear blocks may be used to more closely represent it (see Figure 3).

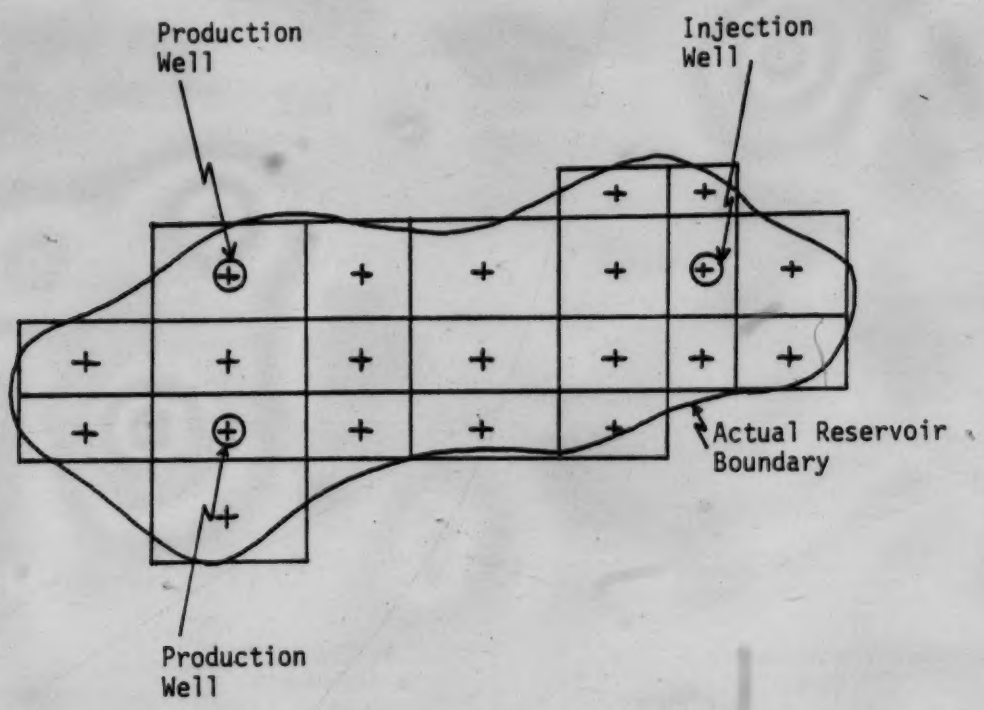


Figure 1. Rectangular Grid Representation of a Petroleum Reservoir

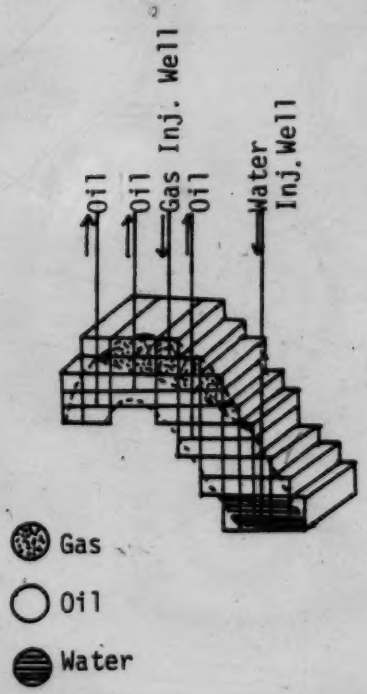
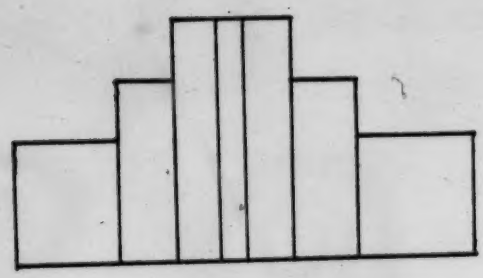


Figure 2. Cross-Sectional Grid Representation of a Petroleum Reservoir



Cross-Sectional View.

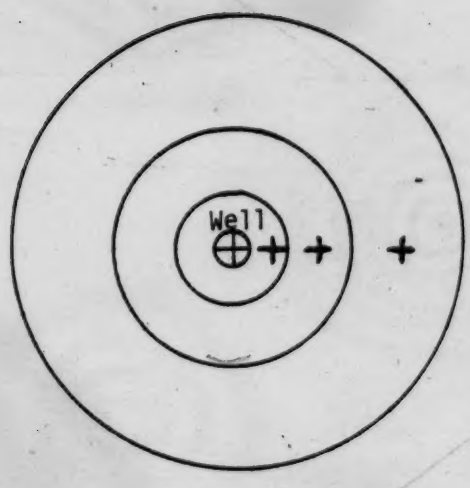


Figure 3. Circular Grid Representation of a Petroleum Reservoir

The partial derivatives in Equations (3.2) are then expressed as algebraic finite difference equations for each grid-block; and after 'linearizing' them by freezing the coefficients over each time step, the resulting system of simultaneous equations can be solved repeatedly to obtain pressures, saturations, and flow rates at the desired advanced time levels and locations. Note that whereas an analytical solution requires the partial differential equations to be satisfied at every point in the domain, results of the finite difference approach are limited to pre-defined points in space and time, and subject to errors which depend on the magnitudes of the chosen increments, as well as subject to truncation and roundoff errors incurred by the finite word length of the computer.

The accuracy and effectiveness of the reservoir model is obviously improved if the number of grid-blocks is increased (and a better definition of the reservoir is provided). But a larger number of blocks rapidly increases computational difficulties; and given the heavy demands on machine time, it is essential to utilize the most efficient solution techniques. A single simulation may involve the solution of several hundred simultaneous equations up to several thousand times.

Within the general framework of the finite difference approach, there exist two basic iterative methods of obtaining a numerical solution to the multiphase flow problem. The first of these, used in the present study, is the so-called 'implicit pressure-explicit saturation' method (hereafter referred to as the 'IMPES'). In this, all variables except one of the phase pressures (usually the oil pressure) are eliminated by use of the phase, saturation and capillary pressure

equations. The resulting second-order equation, containing the unknown oil pressure, is then applied to each grid-block, and the system of equations thus obtained is solved for the oil pressure at the advanced time level for all blocks. Thereafter, rearrangement of the original partial differential equations for the oil and water phases yields the new oil and water saturations at each block. The water and gas phase pressures are obtained from the capillary pressure equations, and gas saturations are calculated from the saturation constraint equation.

The other presently available solution method of which many variations exist, is known as the 'implicit' or 'simultaneous solution' method. This uses the saturation and capillary pressure equations to eliminate saturation terms from the right hand side of the three partial differential phase equations. The resulting three equations, which contain the phase pressures at the advanced time level as unknowns, are then written for each grid-block, and solved simultaneously. Note that while the implicit method is more stable (i.e., allows the use of a larger time increment) than the IMPES method, it requires a three times larger number of equations to be solved, and hence greatly increases the demand for computer time. (Chappelear & Rogers, 1973; Weinstein et al. 1970).

Derivation of the Numerical Model

In order to translate the partial differential equations derived in Equation (3.2) to their algebraic finite difference analogues, it is convenient to rewrite them as follows:

Oil phase

$$\frac{\partial}{\partial x}\left(\frac{A_x k_x k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial x}\right)\cdot\Delta x + \frac{\partial}{\partial y}\left(\frac{A_y k_y k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial y}\right)\cdot\Delta y + \frac{\partial}{\partial z}\left(\frac{A_z k_z k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial z}\right)\cdot\Delta z + q_o^*$$

$$= V_b \frac{\partial}{\partial t}\left(\phi \frac{S_o}{B_o}\right) \quad , \qquad (3.7a)$$

Water phase

$$\frac{\partial}{\partial x}\left(\frac{A_x k_x k_{rw}}{\mu_w B_w}\frac{\partial \Phi_w}{\partial x}\right)\cdot\Delta x + \frac{\partial}{\partial y}\left(\frac{A_y k_y k_{rw}}{\mu_w B_w}\frac{\partial \Phi_w}{\partial y}\right)\cdot\Delta y + \frac{\partial}{\partial z}\left(\frac{A_z k_z k_{rw}}{\mu_w B_w}\frac{\partial \Phi_w}{\partial z}\right)\cdot\Delta z + q_w^*$$

$$= V_b \frac{\partial}{\partial t}\left(\phi \frac{S_w}{B_w}\right) \quad , \qquad (3.7b)$$

Gas phase

$$\frac{\partial}{\partial x}\left(R_{so}\frac{A_x k_x k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial x} + R_{sw}\frac{A_x k_x k_{rw}}{\mu_w B_w}\frac{\partial \Phi_w}{\partial x} + \frac{A_x k_x k_{rg}}{\mu_g B_g}\frac{\partial \Phi_g}{\partial x}\right)\cdot\Delta x$$

$$+ \frac{\partial}{\partial y}\left(R_{so}\frac{A_y k_y k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial y} + R_{sw}\frac{A_y k_y k_{rw}}{\mu_w B_w}\frac{\partial \Phi_w}{\partial y} + \frac{A_y k_y k_{rg}}{\mu_g B_g}\frac{\partial \Phi_g}{\partial y}\right)\cdot\Delta y$$

$$+ \frac{\partial}{\partial z}\left(R_{so}\frac{A_z k_z k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial z} + R_{sw}\frac{A_z k_z k_{rw}}{\mu_w B_w}\frac{\partial \Phi_w}{\partial z} + \frac{A_z k_z k_{rg}}{\mu_g B_g}\frac{\partial \Phi_g}{\partial z}\right)\cdot\Delta z$$

$$+ R_{so} q_o^* + R_{sw} q_w^* + q_g^* = V_b \frac{\partial}{\partial t}\left[\phi\left(R_{so}\frac{S_o}{B_o} + R_{sw}\frac{S_w}{B_w} + \frac{S_g}{B_g}\right)\right] \quad , \qquad (3.7c)$$

where the bulk volume $V_b = \Delta x \Delta y \Delta z$, and $A_x = \Delta y \Delta z$, $A_y = \Delta x \Delta z$, $A_z = \Delta x \Delta y$.

The simplest method of ordering blocks in a three-dimensional grid, such as has been used in the present study, identifies each cell by the subscripts i, j, and k (increasing in the normal x-, y- and z-directions, respectively). Figures 4 and 5 illustrate this ordering scheme.

Applying finite difference approximation techniques to Equations (3.7) then yields:

Oil phase

$$\Delta_x(T^n_{o_x}\Delta_x p_o^{n+1} - T'^n_{o_x}\Delta_x D) + \Delta_y(T^n_{o_y}\Delta_y p_o^{n+1} - T'^n_{o_y}\Delta_y D)$$

$$+ \Delta_z(T^n_{o_z}\Delta_z p_o^{n+1} - T'^n_{o_z}\Delta_z D) + q^*_o = V_b\Delta_t(\phi\frac{S_o}{B_o}), \qquad (3.8a)$$

Water phase

$$\Delta_x(T^n_{w_x}\Delta_x p_w^{n+1} - T'^n_{w_x}\Delta_x D) + \Delta_y(T^n_{w_y}\Delta_y p_w^{n+1} - T'^n_{w_y}\Delta_y D)$$

$$+ \Delta_z(T^n_{w_z}\Delta_z p_w^{n+1} - T'^n_{w_z}\Delta_z D) + q^*_w = V_b\Delta_t(\phi\frac{S_w}{B_w}), \qquad (3.8b)$$

Gas phase

$$\Delta_x(T^n_{og_x}\Delta_x p_o^{n+1} - T'^n_{og_x}\Delta_x D + T^n_{wg_x}\Delta_x p_w^{n+1} - T'^n_{wg_x}\Delta_x D + T^n_{g_x}\Delta_x p_g^{n+1} - T'^n_{g_x}\Delta_x D)$$

$$+ \Delta_y(T^n_{og_y}\Delta_y p_o^{n+1} - T'^n_{og_y}\Delta_y D + T^n_{wg_y}\Delta_y p_w^{n+1} - T'^n_{wg_y}\Delta_y D + T^n_{g_y}\Delta_y p_g^{n+1} - T'^n_{g_y}\Delta_y D)$$

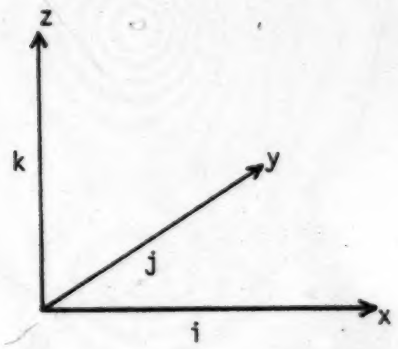


Figure 4. Conventional Identification of Blocks in Three-Dimensional Grid

	(1,4,k) +	(2,4,k) +	(3,4,k) +
j=3	(1,3,k) +	(2,3,k) +	(3,3,k) +
j=2	(1,2,k) +	(2,2,k) +	(3,2,k) +
j=1	(1,1,k) +	(2,1,k) +	(3,1,k) +
	i=1	i=2	i=3

Figure 5. Numbering System For Blocks in Three-Dimensional Grid.

$$+ \Delta_z(T^n_{og_z}\Delta_z p_o^{n+1} - T'^n_{og_z}\Delta_z D + T^n_{wg_z}\Delta_z p_w^{n+1} - T'^n_{wg_z}\Delta_z D + T^n_{g_z}\Delta_z p_g^{n+1} - T'^n_{g_z}\Delta_z D)$$

$$+ R_{so}q_o^* + R_{sw}q_w^* + q_g^* = V_b\Delta_t\left[\phi\left(R_{so}\frac{S_o}{B_o} + R_{sw}\frac{S_w}{B_w} + \frac{S_g}{B_g}\right)\right] , \quad (3.8c)$$

where the difference operator (in any direction r, any block b) is defined by

$$\Delta_r(\alpha_r\Delta_r\beta) \equiv \alpha_{b+1/2}(\beta_{b+1}-\beta_b) - \alpha_{b-1/2}(\beta_b-\beta_{b-1}) \ .$$

Terms that account for the relative ease of fluid flow between grid-blocks are given by the 'transmissibilities'. For example

$$T^n_{o_{i+1/2,j,k}} = \frac{2A_{x_{i,j,k}}A_{x_{i+1,j,k}}k_{x_{i,j,k}}k_{x_{i+1,j,k}}}{A_{x_{i,j,k}}k_{x_{i,j,k}}\Delta x_{i+1,j,k} + A_{x_{i+1,j,k}}k_{x_{i+1,j,k}}\Delta x_{i,j,k}} \times \left.\frac{k_{ro}}{\mu_o B_o}\right|^n_{i+1/2,j,k}$$

where, since $\mu_o = \mu_o(p_o)$, $B_o = B_o(p_o)$,

$$\mu_o B_o|_{i+1/2,j,k} = \mu_o(p_{o_{i+1/2,j,k}})\cdot B_o(p_{o_{i+1/2,j,k}}) \ ,$$

$$p_{o_{i+1/2,j,k}} = \frac{1}{2}(p^n_{o_{i+1,j,k}} + p^n_{o_{i,j,k}}) \ ,$$

and

$$k^n_{ro_{i+1/2,j,k}} = \begin{cases} k^n_{ro_{i+1,j,k}} & \text{if flow is from block i+1 to i} \\ & (\text{viz. } \Phi^n_{i+1,j,k} > \Phi^n_{i,j,k}) \\ k^n_{ro_{i,j,k}} & \text{if flow is from block i to i+1} \\ & (\text{viz. } \Phi^n_{i,j,k} > \Phi^n_{i+1,j,k}) \end{cases}$$

Also, for notational convenience, the symbols "'" and "g" are used to append transmissibility terms. For example,

$$T'_{o_x} = \rho_o g T_{o_x} ,$$

and

$$T_{og_x} = R_{so} T_{o_x} .$$

To implement the IMPES solution method, the three phase equations (Equations (3.8)) are combined, with the aid of the saturation and capillary pressure equations (Equations (3.3) - (3.5)), in order to obtain a single equation in the unknown oil pressure.

Consider, first, the expansion of the right hand side of the oil phase equation (Equation (3.8a)):

$$V_b \Delta_t \left(\phi \frac{S_o}{B_o}\right) = \frac{V_b}{\Delta t}\left\{\left(\phi \frac{S_o}{B_o}\right)^{n+1} - \left(\phi \frac{S_o}{B_o}\right)^n\right\}$$

$$= \frac{V_b}{\Delta t}\left\{S_o^{n+1}\left[\left(\frac{\phi}{B_o}\right)^{n+1} - \left(\frac{\phi}{B_o}\right)^n\right] + \left(\frac{\phi}{B_o}\right)^n \left(S_o^{n+1} - S_o^n\right)\right\}$$

$$= V_b\left\{S_o^{n+1} \Delta_t \left(\frac{\phi}{B_o}\right) + \left(\frac{\phi}{B_o}\right)^n \Delta_t S_o\right\}$$

$$= V_b\left\{S_o^{n+1}\left[\phi^{n+1} \Delta_t \left(\frac{1}{B_o}\right) + \left(\frac{1}{B_o}\right)^n \Delta_t \phi\right] + \left(\frac{\phi}{B_o}\right)^n \Delta_t S_o\right\}$$

$$= \frac{V_b}{\Delta t}\left\{S_o^{n+1} \phi^{n+1} \left(\frac{1}{B_o}\right)' \left(p_o^{n+1} - p_o^n\right) + \frac{S_o^{n+1}}{B_o^n} \phi' \left(p_o^{n+1} - p_o^n\right)\right.$$

$$+ \frac{\phi^n}{B_o^n}(S_o^{n+1}-S_o^n)\} \quad , \qquad (3.9a)$$

where the chord slope "'" is defined as

$$(\alpha)' \equiv \frac{\alpha^{n+1}-\alpha^n}{p^{n+1}-p^n} \quad .$$

Similarly, for the right hand side of the water phase equation (Equation (3.8b)),

$$V_b\Delta_t(\phi \frac{S_w}{B_w}) = \frac{V_b}{\Delta t}\{S_w^{n+1}\phi^{n+1}(\frac{1}{B_w})'(p_w^{n+1}-p_w^n) + \frac{S_w^{n+1}}{B_w^n}\phi'(p_o^{n+1}-p_o^n) + \frac{\phi^n}{B_w^n}(S_w^{n+1}-S_w^n)\} \qquad (3.9b)$$

and, since

$$(\frac{\alpha}{\beta})' = \alpha^{n+1}(\frac{1}{\beta})' + (\frac{1}{\beta})^n\alpha' \quad ,$$

the right hand side of the gas phase equation (Equation (3.8c)) becomes

$$V_b\Delta_t[\phi(R_{so}\frac{S_o}{B_o} + R_{sw}\frac{S_w}{B_w} + \frac{S_g}{B_g})]$$

$$= \frac{V_b}{\Delta t}\{S_o^{n+1}\phi^{n+1}[R_{so}^{n+1}(\frac{1}{B_o})' + \frac{1}{B_o^n}R'_{so}](p_o^{n+1}-p_o^n)$$

$$+ R_{so}^n\frac{S_o^{n+1}}{B_o^n}\phi'(p_o^{n+1}-p_o^n) + R_{so}^n\frac{\phi^n}{B_o^n}(S_o^{n+1}-S_o^n)$$

$$+ S_w\phi^{n+1}[R_{sw}^{n+1}(\frac{1}{B_w})' + \frac{1}{B_w^n}R'_{sw}](p_w^{n+1}-p_w^n)$$

$$+ R_{sw}^n \frac{S_w^{n+1}}{B_w^n} \phi' (p_o^{n+1} - p_o^n) + R_{sw}^n \frac{\phi^n}{B_w^n} (S_w^{n+1} - S_w^n)$$

$$+ S_g^{n+1} \phi^{n+1} \left(\frac{1}{B_g}\right)' (p_g^{n+1} - p_g^n) + \frac{S_g^{n+1}}{B_g^n} \phi' (p_o^{n+1} - p_o^n)$$

$$+ \frac{\phi^n}{B_g^n} (S_g^{n+1} - S_g^n) \} \quad . \qquad (3.9c)$$

Observe that all terms in these expansions are taken at any grid-block with index (i,j,k).

Substitution of Equations (3.4) and (3.5) into the above eliminates dependence on water and gas phase pressures at the n time level. To eliminate saturation terms at the n time level, Equations (3.8) combined with the expansions (3.9) are rearranged so that the right hand sides only contain terms in $(S^{n+1} - S^n)$. The resulting right hand side expressions for each phase equation are then

Oil phase

$$\frac{\phi^n}{B_o^n} (S_o^{n+1} - S_o^n) \quad ,$$

Water phase

$$\frac{\phi^n}{B_w^n} (S_w^{n+1} - S_w^n) \quad ,$$

Gas phase

$$R_{so}^n \frac{\phi^n}{B_o^n}(S_o^{n+1}-S_o^n) + R_{sw}^n \frac{\phi^n}{B_w^n}(S_w^{n+1}-S_w^n) + \frac{\phi^n}{B_g^n}(S_g^{n+1}-S_g^n) \quad .$$

Multiplying the oil phase equation (Equation (3.8a)) by

$$\frac{B_o^n}{B_g^n} - R_{so}^n \quad ,$$

the water phase equation (Equation (3.8b)) by

$$\frac{B_w^n}{B_g^n} - R_{sw}^n \quad ,$$

summing together with Equation (3.8c), and applying Equation (3.3) causes the right hand side terms to vanish:

$$\left(\frac{B_o^n}{B_g^n} - R_{so}^n\right)\frac{\phi^n}{B_o^n}(S_o^{n+1}-S_o^n) + \left(\frac{B_w^n}{B_g^n} - R_{sw}^n\right)\frac{\phi^n}{B_w^n}(S_w^{n+1}-S_w^n)$$

$$+ R_{so}^n \frac{\phi^n}{B_o^n}(S_o^{n+1}-S_o^n) + R_{sw}^n(S_w^{n+1}-S_w^n) + \frac{\phi^n}{B_g^n}(S_g^{n+1}-S_g^n)$$

$$= \frac{\phi^n}{B_g^n}\{(S_o^{n+1}+S_w^{n+1}+S_g^{n+1}) - (S_o^n+S_w^n+S_g^n)\} = 0 \quad .$$

Since calculating the changes in oil pressures rather than the oil pressures themselves reduces round-off error, define

$$\delta p_{o_{i,j,k}} \equiv p_{o_{i,j,k}}^{n+1} - p_{o_{i,j,k}}^n \quad ,$$

and, in a notation convenient for computer implementation, the resulting sum of the three phase equations can then be written in the form

$$Z\delta p_{o_{i,j,k-1}} + B\delta p_{o_{i,j-1,k}} + D\delta p_{o_{i-1,j,k}} + E\delta p_{o_{i,j,k}}$$

$$+ F\delta p_{o_{i+1,j,k}} + H\delta p_{o_{i,j+1,k}} + S\delta p_{o_{i,j,k+1}} = q_{i,j,k} \quad , \tag{3.10}$$

where $E = -(Z + B + D + F + H + S + \frac{\Gamma}{\Delta t})$,

$$Z = \left(\frac{B_o^n}{B_g^n} - R_{so}^n \right) T_{o_{z_{i,j,k-1/2}}}^n + \left(\frac{B_w^n}{B_g^n} - R_{sw}^n \right) T_{w_{z_{i,j,k-1/2}}}^n$$

$$+ T_{og_{z_{i,j,k-1/2}}}^n + T_{wg_{z_{i,j,k-1/2}}}^n + T_{g_{z_{i,j,k-1/2}}}^n \quad ,$$

$$B = \left(\frac{B_o^n}{B_g^n} - R_{so}^n \right) T_{o_{y_{i,j-1/2,k}}}^n + \left(\frac{B_w^n}{B_g^n} - R_{sw}^n \right) T_{w_{y_{i,j-1/2,k}}}^n$$

$$+ T_{og_{y_{i,j-1/2,k}}}^n + T_{wg_{y_{i,j-1/2,k}}}^n + T_{g_{y_{i,j-1/2,k}}}^n \quad ,$$

$$D = \left(\frac{B_o^n}{B_g^n} - R_{so}^n \right) T_{o_{x_{i-1/2,j,k}}}^n + \left(\frac{B_w^n}{B_g^n} - R_{sw}^n \right) T_{w_{x_{i-1/2,j,k}}}^n$$

$$+ T_{og_{x_{i-1/2,j,k}}}^n + T_{wg_{x_{i-1/2,j,k}}}^n + T_{g_{x_{i-1/2,j,k}}}^n \quad ,$$

$$F = \left(\frac{B_o^n}{B_g^n} - R_{so}^n \right) T_{o_{x_{i+1/2,j,k}}}^n + \left(\frac{B_w^n}{B_g^n} - R_{sw}^n \right) T_{w_{x_{i+1/2,j,k}}}^n$$

$$+ T^n_{og_{x_{i+1/2,j,k}}} + T^n_{wg_{x_{i+1/2,j,k}}} + T^n_{g_{x_{i+1/2,j,k}}} ,$$

$$H = \left(\frac{B^n_o}{B^n_g} - R^n_{so}\right)T^n_{o_{y_{i,j+1/2,k}}} + \left(\frac{B^n_w}{B^n_g} - R^n_{sw}\right)T^n_{w_{y_{i,j+1/2,k}}}$$

$$+ T^n_{og_{y_{i,j+1/2,k}}} + T^n_{wg_{y_{i,j+1/2,k}}} + T^n_{g_{y_{i,j+1/2,k}}} ,$$

$$S = \left(\frac{B^n_o}{B^n_g} - R^n_{so}\right)T^n_{o_{z_{i,j,k+1/2}}} + \left(\frac{B^n_w}{B^n_g} - R^n_{sw}\right)T^n_{w_{z_{i,j,k+1/2}}}$$

$$+ T^n_{og_{z_{i,j,k+1/2}}} + T^n_{wg_{z_{i,j,k+1/2}}} + T^n_{g_{z_{i,j,k+1/2}}} ,$$

$$\Gamma = V_{b_{i,j,k}}\{S^{n+1}_o \phi^{n+1}\left(\frac{1}{B_o}\right)'\left[\left(R^{n+1}_{so} - R^n_{so}\right) + \frac{B^n_o}{B^n_g}\right]$$

$$+ S^{n+1}_w \phi^{n+1}\left(\frac{1}{B_w}\right)'\left[\left(R^{n+1}_{sw} - R^n_{sw}\right) + \frac{B^n_w}{B^n_g}\right]$$

$$+ S^{n+1}_g \phi^{n+1}\left(\frac{1}{B_g}\right)' + \frac{S^{n+1}_o \phi^{n+1}}{B^n_o} R'_{so} + \frac{S^{n+1}_w \phi^{n+1}}{B^n_w} R'_{sw} + \frac{\phi'}{B^n_g}\} ,$$

and

$$q_{i,j,k} = \Delta_x T^n_{wg_x} \Delta_x p^{n+1}_{cow} + \Delta_y T^n_{wg_y} \Delta_y p^{n+1}_{cow} + \Delta_z T^n_{wg_z} \Delta_z p^{n+1}_{cow}$$

$$- \Delta_x T^n_{g_x} \Delta_x p^{n+1}_{cgo} - \Delta_y T^n_{g_y} \Delta_y p^{n+1}_{cgo} - \Delta_z T^n_{g_z} \Delta_z p^{n+1}_{cgo}$$

$$+ \left(\frac{B^n_o}{B^n_g} - R^n_{so}\right)\left[\Delta_x T'^n_{o_x} \Delta_x D + \Delta_y T'^n_{o_y} \Delta_y D + \Delta_z T'^n_{o_z} \Delta_z D\right.$$

$$\left. - \Delta_x T^n_{o_x} \Delta_x p^n_o - \Delta_y T^n_{o_y} \Delta_y p^n_o - \Delta_z T^n_{o_z} \Delta_z p^n_o\right]$$

$$+ \left(\frac{B^n_w}{B^n_g} - R^n_{sw}\right)\left[\Delta_x T'^n_{w_x} \Delta_x D + \Delta_y T'_{w_y} \Delta_y D + \Delta_z T'^n_{w_z} \Delta_z D\right.$$

$$\left. - \Delta_x T^n_{w_x} \Delta_x p^n_o - \Delta_y T^n_{w_y} \Delta_y p^n_o - \Delta_z T^n_{w_z} \Delta_z p^n_o\right]$$

$$+ \left(\frac{B^n_w}{B^n_g} - R^n_{sw}\right)\left[\Delta_x T^n_{w_x} \Delta_x p^{n+1}_{cow} + \Delta_y T^n_{w_y} \Delta_y p^{n+1}_{cow}\right.$$

$$\left. + \Delta_z T^n_{w_z} \Delta_z p^{n+1}_{cow}\right]$$

$$+ \frac{V_b}{\Delta t}\left[-(S_w \phi R_{sw})^{n+1}\left(\frac{1}{B_w}\right)'(p^{n+1}_{cow} - p^n_{cow}) - \frac{(S_w\phi)^{n+1}}{B^n_w} R'_{sw}(p^{n+1}_{cow} - p^n_{cow})\right.$$

$$\left. + (S_g \phi)^{n+1}\left(\frac{1}{B_g}\right)'(p^{n+1}_{cgo} - p^n_{cgo})\right]$$

$$+ \left(\frac{B^n_w}{B^n_g} - R^n_{sw}\right)\frac{V_b}{\Delta t}\left[-(S_w\phi)^{n+1}\left(\frac{1}{B_w}\right)'(p^{n+1}_{cow} - p^n_{cow})\right]$$

$$+ \Delta_x T'^n_{og_x} \Delta_x D + \Delta_y T'^n_{og_y} \Delta_y D + \Delta_z T'^n_{og_z} \Delta_z D$$

$$+ \Delta_x T'^n_{wg_x} \Delta_x D + \Delta_y T'^n_{wg_y} \Delta_y D + \Delta_z T'^n_{wg_z} \Delta_z D$$

$$+ \Delta_x T'^n_{g_x} \Delta_x D + \Delta_y T'^n_{g_y} \Delta_y D + \Delta_z T'^n_{g_z} \Delta_z D$$

$$- \Delta_x T^n_{og_x} \Delta_x p^n_o - \Delta_y T^n_{og_y} \Delta_y p^n_o - \Delta_z T^n_{og_z} \Delta_z p^n_o$$

$$- \Delta_x T^n_{wg_x} \Delta_x p^n_o - \Delta_y T^n_{wg_y} \Delta_y p^n_o - \Delta_z T^n_{wg_z} \Delta_z p^n_o$$

$$- \Delta_x T^n_{g_x} \Delta_x p^n_o - \Delta_y T^n_{g_y} \Delta_y p^n_o - \Delta_z T^n_{g_z} \Delta_z p^n_o$$

$$- \left(\frac{B^n_o}{B^n_g} - R^n_{so}\right) q^*_o - \left(\frac{B^n_w}{B^n_g} - R^n_{sw}\right) q^*_w - q^*_g - R^n_{so} q^*_o - R^n_{sw} q^*_w \quad .$$

Writing Equation (3.10) for each grid-block, and ordering the grid as in Figure 6, then yields a system of linear equations of the form $\underline{A}\underline{x} = \underline{b}$, where A is the coefficient matrix containing Z, B, D, E, F, H, S (see Figure 7), $\underline{x}$ is the column vector of the unknown δp_o terms, and $\underline{b}$ is the column vector containing the $q_{i,j,k}$ terms. (For alternative ordering schemes, see Price & Coats, 1974).

Once the phase pressures have been calculated for each grid-block, it is possible to calculate the phase saturations explicitly by means of the original difference equations. Rearrangement of the oil and water phase equations (Equations (3.8a) and (3.8b)) yields

$$S^{n+1}_o = \left\{\frac{\Delta t}{V_b}\left[\Delta_x\left(T^n_{o_x} \Delta_x p^{n+1}_o - T'^n_{o_x} \Delta_x D\right) + \Delta_y\left(T^n_{o_y} \Delta_y p^{n+1}_o - T'^n_{o_y} \Delta_y D\right)\right.\right.$$

$$\left.\left. + \Delta_z\left(T^n_{o_z} \Delta_z p^{n+1}_o - T'^n_o \Delta_z D\right) + q^*_o\right] + \frac{\phi^n S^n_o}{B^n_o}\right\}$$

$$\times \left\{\phi^{n+1}\left(\frac{1}{B_o}\right)'\left(p^{n+1}_o - p^n_o\right) + \frac{\phi'}{B^n_o}\left(p^{n+1}_o - p^n_o\right) + \frac{\phi^n}{B^n_o}\right\}^{-1} \quad ,$$

and

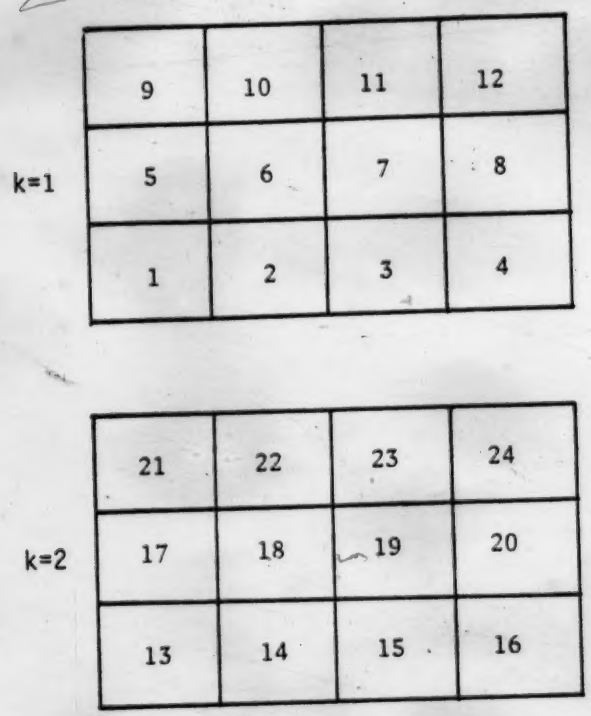


Figure 6. Normal Ordering For Grid System

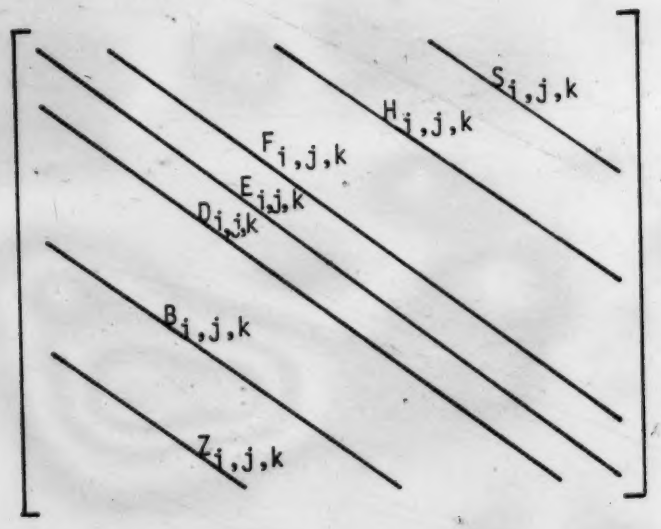


Figure 7. Coefficient Matrix For Normally Ordered Three-Dimensional Grid System

$$S_w^{n+1} = \{\frac{\Delta t}{V_b}[\Delta_x(T^n_{w_x}\Delta_x p_w^{n+1} - T'^n_{w_x}\Delta_x D) + \Delta_y(T^n_{w_y}\Delta_y p_w^{n+1} - T'^n_{w_y}\Delta_y D)$$

$$+ \Delta_z(T^n_{w_z}\Delta_z p_w^{n+1} - T'^n_{w_z}\Delta_z D) + q_w^*] + \frac{\phi^n S_w^n}{B_w^n}\}$$

$$\times \{\phi^{n+1}(\frac{1}{B_w})' [(p_o^{n+1} - p_o^n) - (p_{cow}^{n+1} - p_{cow}^n)] + \frac{\phi'}{B_w^n}(p_o^{n+1} - p_o^n) + \frac{\phi^n}{B_w^n}\}^{-1} \quad .$$

The gas phase saturations can then be determined by means of Equation (3.3).

Since the coefficients of Equation (3.10) are dependent on both the unknown pressures and saturations, the solution process must be carried out by iteration. Initial estimates of pressures, saturations, and production/injection rates are made (usually based on graphical extrapolation schemes), and the resultant calculated values in each grid-block are compared with these estimates. If convergence to a solution (within a prescribed tolerance) has not been obtained, calculations are repeated with these latest values. Alternatively, a material balance is established, and if acceptable, the calculations advance to the next time step. If the material balance is unreasonable, the time step is diminished, and pressures, saturations, and production/injection rates are recalculated. The material balance check, defined as

$$MB \equiv \frac{\text{Net change in mass over time interval}}{\text{Net mass throughput}} = 1 \quad ,$$

is valuable as a test of the principle of conservation of mass, as well as an independent check on the accuracy of the finite difference solution.

Further information on reservoir simulator development can be found in publications by Craft and Hawkins, 1959; Peaceman, 1969; Ford, 1971; Thomas, 1982; and Farouq Ali, 1980.

Computational Procedure

A computer program to solve the numerical model for a petroleum reservoir, as developed in the preceding section, was written and coded in FORTRAN IV language, and run on an Amdahl 470 computer under the MTS operating system. The program is composed of a series of short, well-defined sections, each responsible for solving a specific part of the problem, and contains two subroutines to allow for interpolation of data by cubic splines. The program may be briefly described as follows:

Initially, the program reads in and prints out all required reservoir parameters, time-invariant data and program flags that assign various options to a particular simulation. Thereafter, the constant parts of the transmissibility terms are calculated (with the no-flow condition at the outer boundaries of the reservoir being represented by zero transmissibilities at the boundary points); the initial estimates of fluid properties are made; and pressure and saturation distributions are produced. The first major loop is then set up, and provides for advancement to the next time level when a solution for the present time has been obtained. The remainder of the transmissibility terms, as well as the coefficients Z, B, D, F, H, and S, are subsequently calculated. The second major loop is started to provide for further iteration when convergence to a solution has not been attained. Fluid properties are updated to the extrapolated time level, and the E coefficient and

$q_{i,j,k}$ terms are determined. After solving for the pressure and saturation distributions, a check determines if the solutions have converged. If so, a mass balance check is performed, and if acceptable, the pressure and saturation distributions, together with the production/injection rates and mass balances, are printed out. A time increment is selected; all reservoir properties are updated; pressure and saturation distributions are extrapolated to the next time level; and the program returns to the outer loop. If the tests for convergence or the mass balance checks are unacceptable, the program updates properties to the latest estimates and returns to the inner loop. A maximum number of iterations is prescribed to account for the possibility of an oscillating solution, and if exceeded, the program breaks the loop to print a warning message.

A flow chart of the computer program is presented in Figure 8.

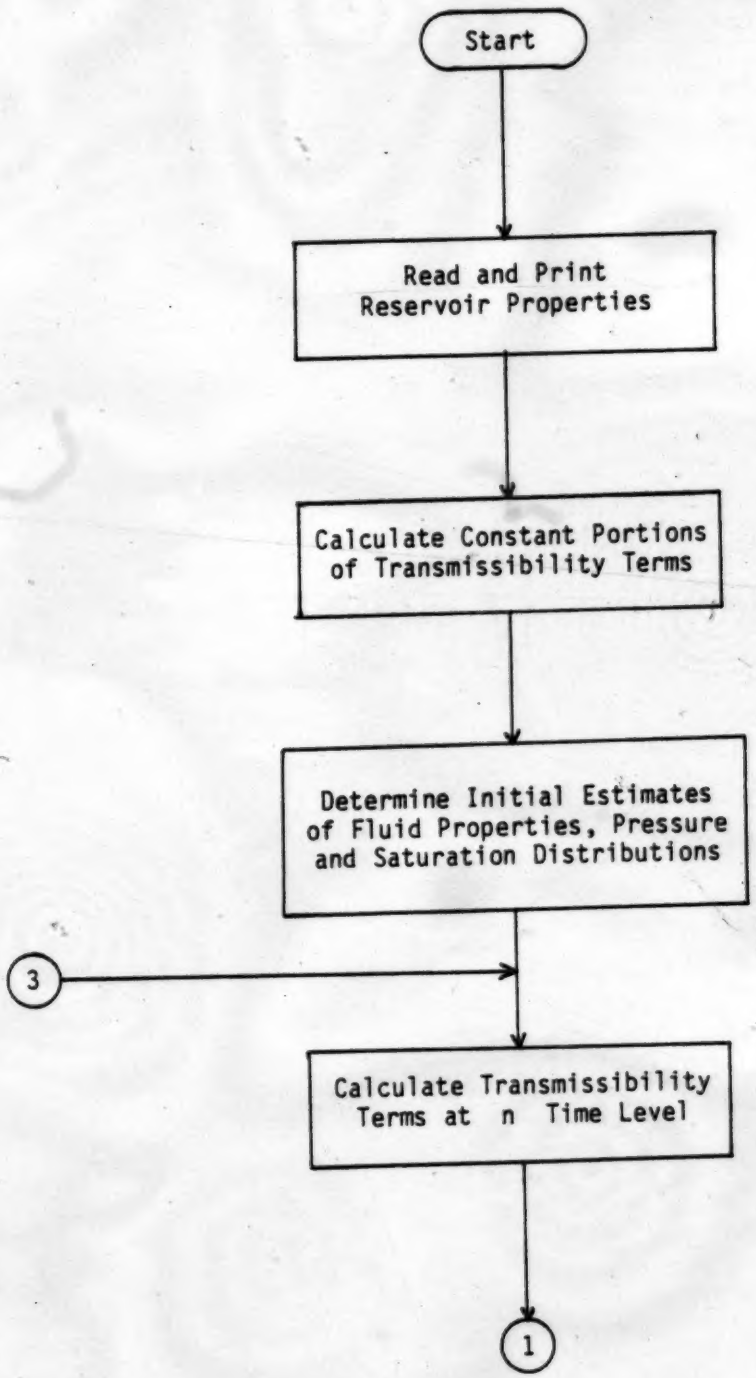

Start
Read and Print Reservoir Properties
Calculate Constant Portions of Transmissibility Terms
Determine Initial Estimates of Fluid Properties, Pressure and Saturation Distributions
3
Calculate Transmissibility Terms at n Time Level
1

Figure 8. Flow Diagram of Computational Procedure

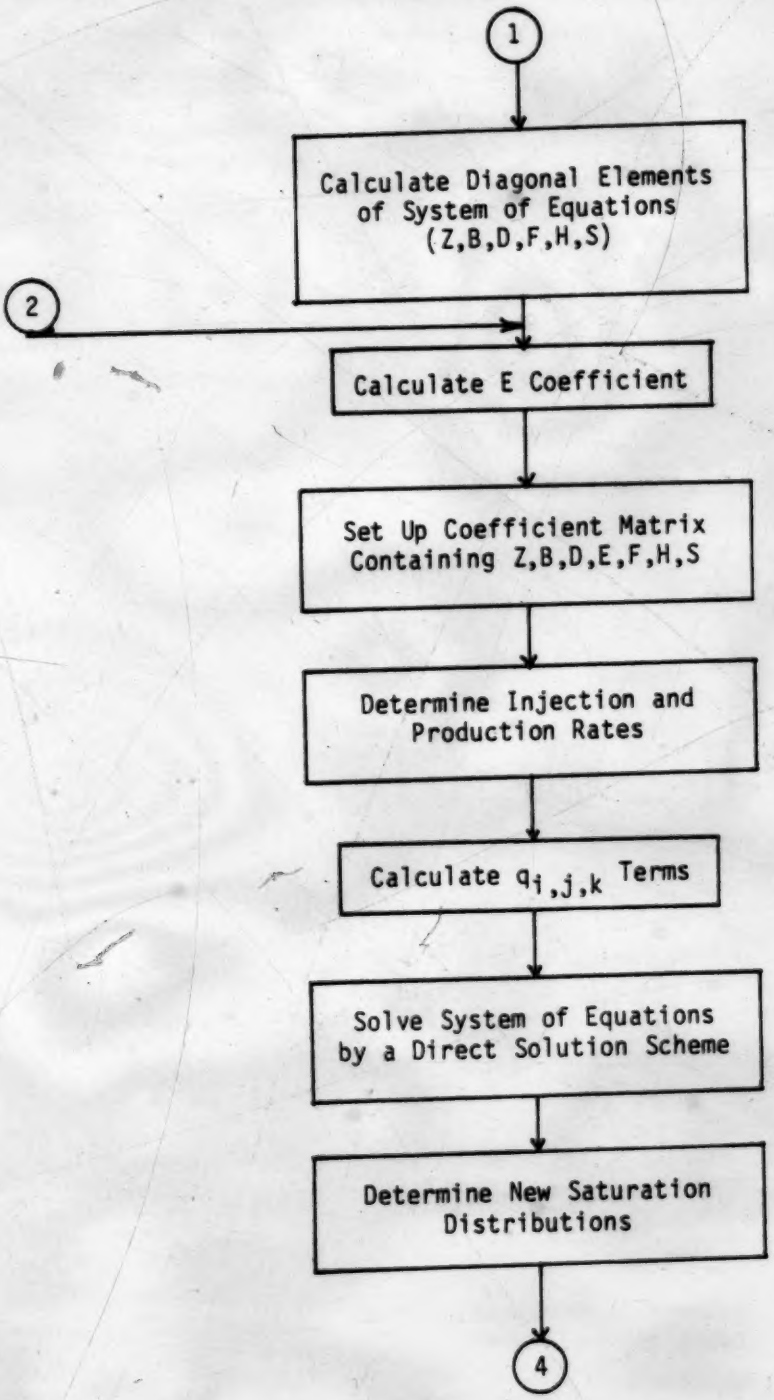

1
Calculate Diagonal Elements of System of Equations (Z,B,D,F,H,S)
2
Calculate E Coefficient
Set Up Coefficient Matrix Containing Z,B,D,E,F,H,S
Determine Injection and Production Rates
Calculate $q_{i,j,k}$ Terms
Solve System of Equations by a Direct Solution Scheme
Determine New Saturation Distributions
4

Figure 8. (Continued)

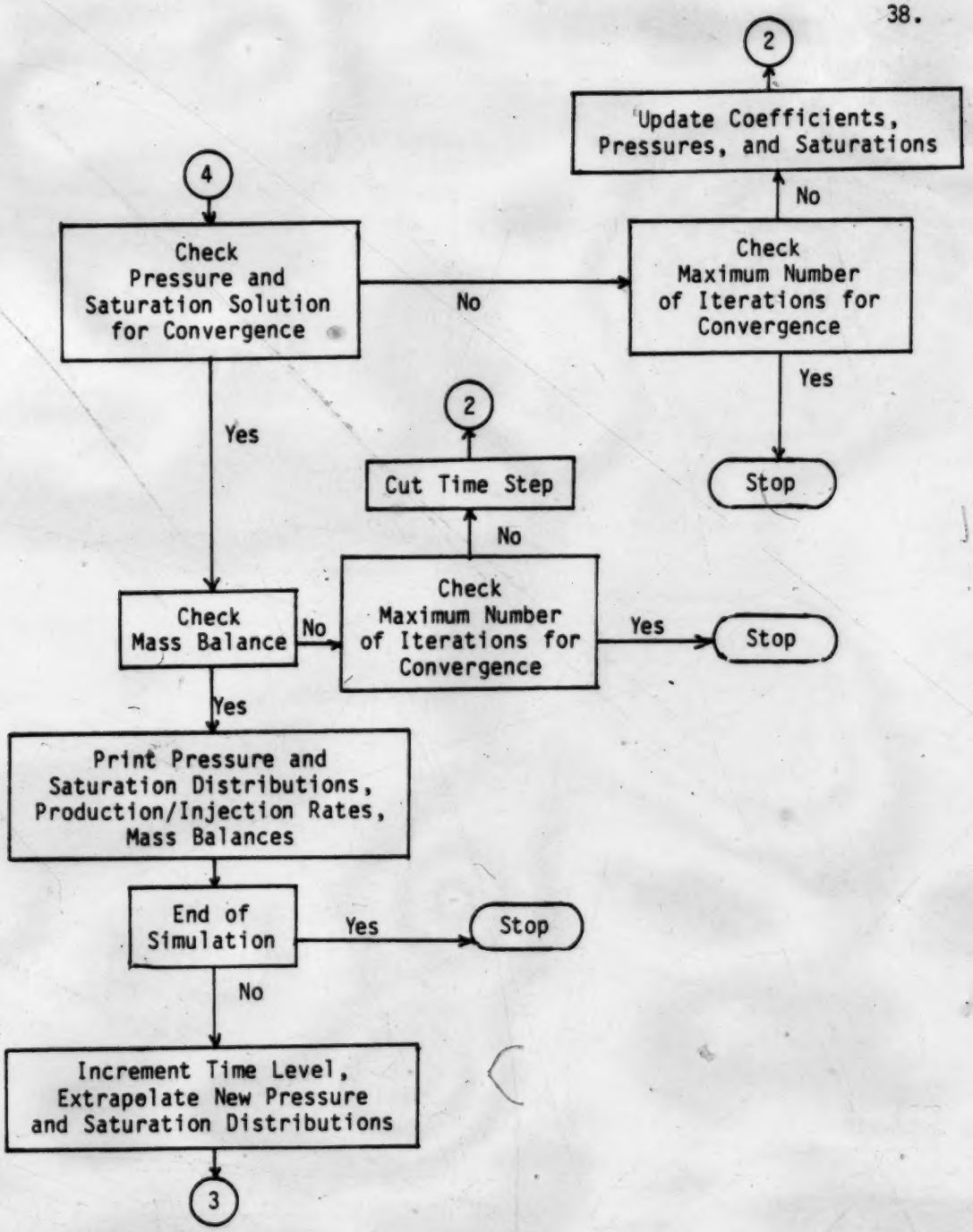


Figure 8. (Continued)

CHAPTER FOUR

LINEAR PROGRAMMING METHODS FOR AUTOMATIC HISTORY-MATCHING

As noted in Chapter 2 (Review of the Literature), the first automatic history-matching algorithm that yielded feasible estimates of reservoir properties for practical field situations was developed by Coats et al. (1970), who employed a combination of least-squares and linear programming methods. While more complex automatic history-matching algorithms, often based on linear programming, were designed in later years, these did not significantly improve the original technique (Slater & Durrer, 1971; Thomas et al., 1972; Carter et al. 1974); and nonlinear optimization techniques could not guarantee attainment of a globally minimum matching error (Wasserman et al., 1975). Furthermore, although the system of partial differential equations, which constitutes the reservoir model is basically nonlinear, studies by Carter et al. (1970) support the contention of Coats et al. (1970) that the matching error-parameter relationship could be adequately approximated by a linear function, especially where reservoirs possess relatively high porosities and permeabilities (Slater & Durrer, 1971).

Since the work of Coats et al. (1970) has attracted much attention - and most authors who considered it obtained supporting evidence and acknowledged the technique as at least a basically practical approach to automatic history-matching - it is pertinent to analyze the algorithm further in order to determine its validity more precisely.

Derivation of the Algorithm

As developed by Coats et al. (1970), the least-squares, linear programming algorithm for automatic history-matching is derived as follows:

Let x_j $(j=1,2,\ldots,J)$ represent the porosity and permeability parameters that constitute the reservoir description data. In order to reduce the number of these unknowns, the reservoir is partitioned into zones (based on existing geological data), which are each assumed to possess constant porosity and permeability. Now, let d_i $(i=1,2,\ldots,I)$ represent performance data (pressure distributions, production/injection rates), with observed performance data denoted by d_i^{obs} and calculated data (from simulator runs employing various values of x_j) denoted by d_i^{calc}. The matching error set, ε_i, is defined by

$$\varepsilon_i \equiv d_i^{obs} - d_i^{calc} \quad ,$$

and the history-matching problem lies in determining a set of description parameters, x_j^*, which will minimize some norm of this error set. One notes here that, in order to utilize optimization techniques, the number of performance data must exceed the number of reservoir description parameters, i.e., $I > J$. For each x_j to be estimated, it is also necessary to impose upper and lower bounds, denoted x_{ju} and x_{jl}, respectively (and based on existing geological data), such that $0 < x_{jl} < x_j < x_{ju}$.

A total of N simulations are then performed, each using a different set of random description parameters, x_j. These random parameters, denoted x_j^r for each run r ($r=1,2,\ldots,N$), generate the calculated performance data, $d_i^{calc,r}$, and the resulting matching errors, ε_i^r. For any run r, each x_j^r is selected using a uniform random number generator, so that

$$x_j^r = x_{j1} + R(x_{ju} - x_{j1})$$

where R is a random number between 0 and 1.

On the supposition that the matching error, ε_i, is a single-valued function of the reservoir description parameters, Coats et al. (1970) chose to model this dependence by a linear relationship, and justified the approximation by pointing to the reasonable success of the resultant history-matching algorithm. They thus defined the relation between the matching errors and reservoir description parameters as

$$\varepsilon_i \equiv \sum_{j=0}^{J} a_{ij} x_j \quad , \qquad (4.1)$$

where $i = 1,2,\ldots,I$, and $x_0 \equiv 1$.

A procedure for determining the unknown a_{ij} in Equation (4.1) was then developed by making use of a least-squares approach. For each run r, the deviation D_i^r is defined as

$$D_i^r \equiv \varepsilon_i^r - \sum_{j=0}^{J} a_{ij} x_j^r \quad ,$$

where $\varepsilon_i^r = d_i^{obs} - d_i^{calc,r}$, $i = 1,2,\ldots,I$, and $r = 1,2,\ldots,N$. The sum $\sum_{r=1}^{N} (D_i^r)^2$ is then minimized (differentiate and set to zero) to yield

$$\sum_{j=0}^{J} \left[\sum_{r=1}^{N} x_n^r x_j^r \right] a_{ij} = \sum_{r=1}^{N} \varepsilon_i^r x_n^r , \qquad (4.2)$$

where $i = 1,2,\ldots,I$, and $n = 0,1,2,\ldots,J$. An alternative derivation of Equation (4.2) is provided in the Appendix.

For each i, Equation (4.2) represents $J + 1$ simultaneous linear equations in the $J + 1$ unknowns $a_{i0}, a_{i1}, \ldots, a_{iJ}$. It is helpful to expand Equation (4.2), for any i, by writing

$$n = 0: \quad \left(\sum_{r=1}^{N} x_0^r x_0^r \right) a_{i0} + \left(\sum_{r=1}^{N} x_0^r x_1^r \right) a_{i1} + \ldots + \left(\sum_{r=1}^{N} x_0^r x_J^r \right) a_{iJ} = \sum_{r=1}^{N} \varepsilon_i^r x_0^r$$

$$n = 1: \quad \left(\sum_{r=1}^{N} x_1^r x_0^r \right) a_{i0} + \left(\sum_{r=1}^{N} x_1^r x_1^r \right) a_{i1} + \ldots + \left(\sum_{r=1}^{N} x_1^r x_J^r \right) a_{iJ} = \sum_{r=1}^{N} \varepsilon_i^r x_1^r$$

- - - - - - - - -

$$n = J: \quad \left(\sum_{r=1}^{N} x_J^r x_0^r \right) a_{i0} + \left(\sum_{r=1}^{N} a_J^r x_1^r \right) a_{i1} + \ldots + \left(\sum_{r=1}^{N} x_J^r x_J^r \right) a_{iJ} = \sum_{r=1}^{N} \varepsilon_i^r x_J^r .$$

Note that the coefficients of the a_{ij} terms are constant with respect to the index i, and that this method essentially decouples the equations for each ε_i from those of every other ε_i .

To solve for the unknown a_{ij}, define $\underline{a}_i$ and $\underline{c}_i$ as the column vectors $(a_{i0}, a_{i1}, \ldots, a_{iJ})'$ and $(c_{i0}, c_{i1}, \ldots, c_{iJ})'$, respectively, and

the matrix B containing the b_{nj} coefficients, where

$$b_{nj} \equiv \sum_{r=1}^{N} x_n^r x_j^r \quad , \quad c_{in} \equiv \sum_{r=1}^{N} \varepsilon_i^r x_n^r \quad ,$$

for $i = 1,2,\ldots,I$, and $n = 0,1,2,\ldots,J$.

Then

$$\underline{a}_i = B^{-1}\underline{c}_i \quad , \tag{4.3}$$

and repeated application of Equation (4.3) for $i = 1,2,\ldots,I$ yields the entire set of a_{ij}. Since the b_{nj} coefficients are independent of i, the matrix B need be calculated only once and stored. Note that if the matrix B is found to be non-invertible, it is necessary to rerun the random number generator until an acceptable set of values is obtained.

Equation (4.1) gives

$$\varepsilon_i = \sum_{j=0}^{J} a_{ij} x_j \quad ,$$

where the a_{ij} are now known, and the inverse problem of determining a best set of reservoir description paramters, x_j^*, is therefore reduced to minimizing some norm of the errors subject to the constraints $0 < x_{jl} < x_j < x_{ju}$. Coats et al. (1970) chose to minimize

$$S \equiv \sum_{i=1}^{I} |\varepsilon_i| \quad , \tag{4.4}$$

with the goal of eliminating negative porosity and permeability estimates.

In order to utilize a linear programming solution technique for minimizing Equation (4.4), ε_i is expressed in terms of slack variables:

$$\varepsilon_i = x_{J+I+i} - x_{J+i} \quad , \quad i = 1,2,\ldots,I \quad .$$

Combining this with Equation (4.1) then yields

$$a_{io} + x_{J+i} - x_{J+I+i} + \sum_{j=1}^{J} a_{ij} x_j = 0 \quad ,$$

where $i = 1,2,\ldots,I$. Also, Equation (4.4) becomes

$$S = \sum_{i=1}^{I} |x_{J+I+i} - x_{J+i}| \quad ,$$

and by the triangle inequality, an upper bound on S is minimized when

$$\sum_{i=1}^{I} x_{J+I+i} + \sum_{i=1}^{I} x_{J+i}$$

attains a minimum value. Note that, since

$$\left| \sum_{i=1}^{I} x_{J+I+i} - \sum_{i=1}^{J} x_{J+i} \right| \leq S \leq \sum_{i=1}^{I} x_{J+I+i} + \sum_{i=1}^{I} x_{J+i} \quad ,$$

upper and lower bound for S can be computed. The linear programming problem can consequently be formulated as follows:

$$\text{Minimize} \quad \sum_{i=1}^{I} x_{J+I+i} + \sum_{i=1}^{I} x_{J+i}$$

subject to the constraints

$$\text{(i)}\quad a_{i0} + x_{J+i} - x_{J+I+i} + \sum_{j=1}^{J} a_{ij}x_j = 0 \quad , \quad i = 1,2,\ldots,I$$

$$\text{(ii)}\quad x_j - x_{ju} + x_{J+2I+j} = 0 \quad , \quad j = 1,2,\ldots,J$$

$$\text{(iii)}\quad x_{jl} - x_j + x_{2J+2I+j} = 0 \quad , \quad j = 1,2,\ldots,J$$

$$\text{(iv)}\quad x_j \geq 0 \quad , \quad j = 1,2,\ldots,2I + 3J \ .$$

The total of $2I + 3J$ variables are defined as

Reservoir description parameters:

$$x_j \ , \quad j = 1,2,\ldots,J$$

Slack variables for errors, ε_i :

$$x_{J+1},\ x_{J+2},\ldots,\ x_{J+I},\ x_{J+I+1},\ x_{J+I+2},\ldots,\ x_{J+2I}$$

Slack variables for upper bound constraints on the x_j:

$$x_{J+2I+1},\ x_{J+2I+2},\ldots,\ x_{2J+2I}$$

Slack variables for lower bound constraints on the x_j:

$$x_{2J+2I+1},\ x_{2J+2I+2},\ldots,\ x_{3J+2I} \ .$$

Solution by the simplex method, which then furnishes a best set of estimates for the reservoir description parameters, is straightforward.

Computational Procedure

A computer program for implementing the automatic history-matching algorithm of Coats et al. (1970) was written, coded in FORTRAN IV language, and run on an Amdahl 470 computer under the MTS operating system. As the algorithm required a petroleum reservoir simulator, the program was designed to be compatible with the simulator developed in Chapter 3.

The program reads the initial reservoir data, including the observed performance data, d_i^{obs}, and the constraints on the description parameters to be estimated. A loop, enclosing the entire simulator, then uses a random number generator to select the reservoir porosities and permeabilities, x_j^r, and calculates the performance data, $d_i^{calc,r}$, and the errors, ε_i^r, for a specified number of runs. Using the description parameters, x_j^r, and the errors ε_i^r, the matrix B and vector $\underline{c}_i$ are then set up, and the a_{ij} coefficients are calculated. Thereafter, the coefficient matrix of the constraint equations for the linear programming problem is determined, and solution by the simplex method subsequently yields the best estimates of the reservoir description properties.

A flow chart of the computer program is presented in Figure 9.

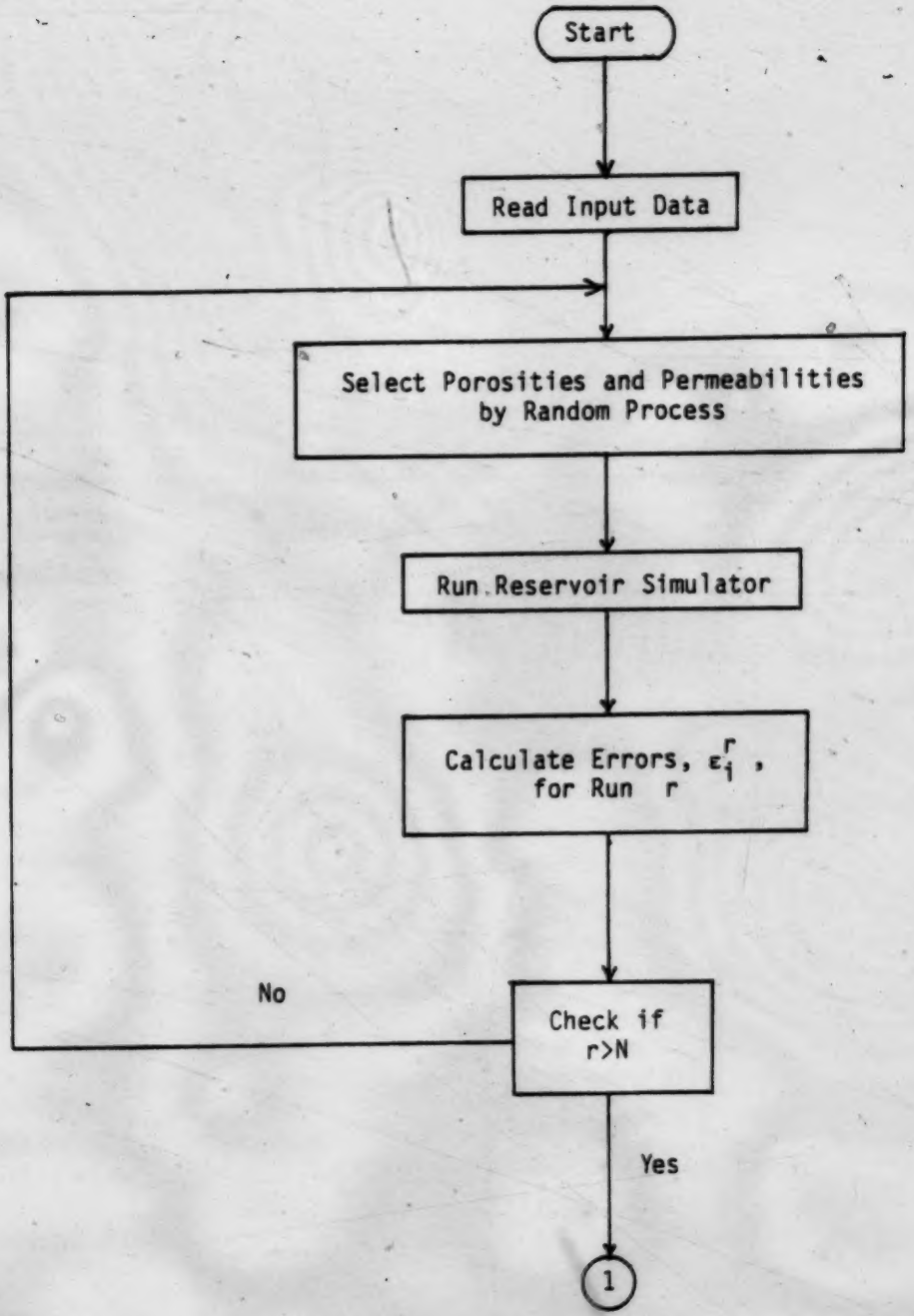


Figure 9. Flow Diagram of Computational Procedure

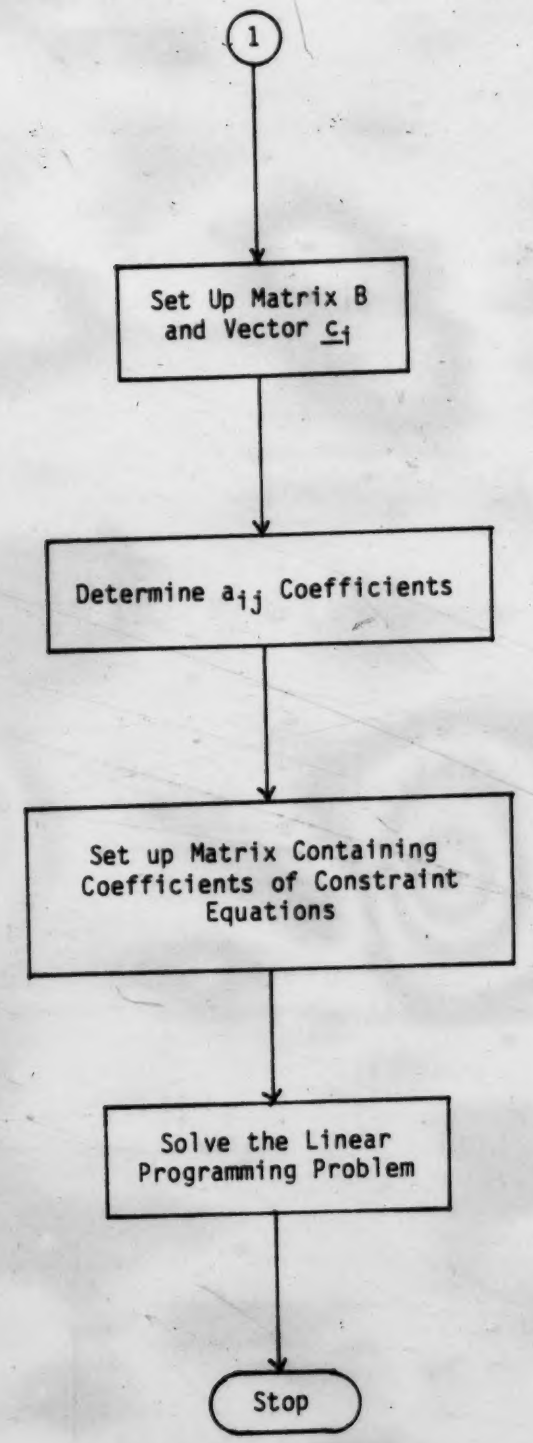

1
Set Up Matrix B and Vector $\underline{c}_i$
Determine a_{ij} Coefficients
Set up Matrix Containing Coefficients of Constraint Equations
Solve the Linear Programming Problem
Stop

Figure 9. (Continued)

Discussion

Although the experimental results reported by Coats et al. (1970), Slater & Durrer (1971), Thomas et al. (1972), and Carter et al. (1974) indicate that the automatic history-matching method proposed by Coats et al. (1970) produces reasonable estimates of reservoir description parameters, it is pertinent to examine the theoretical foundation of the algorithm.

Without benefit of an experimental test of the algorithm, the assumed linear functional dependence of matching errors on reservoir description parameters, and the subsequent reasoning for deriving the automatic history-matching procedure, may, a priori, be expected to yield poor results.

Consider, first, the method by which the a_{ij} coefficients are determined. In least-squares calculations, the assumption of an approximate linear behaviour implies that

$$d^{calc}(\alpha \underline{x}) \approx \alpha d^{calc}(\underline{x})$$

where α is a constant, $\underline{x}$ is a set of reservoir description parameters, and $d^{calc}(\underline{x})$ are the calculated performance data based on the description parameters provided. For the complex system of nonlinear partial differential equations that describe the flow mechanics of a petroleum reservoir, such simple linear dependence would, intuitively, appear unlikely. That experimental testing of numerous reservoir simulators under various practical field situations has shown the approximation and resulting linear programming solution to be generally valid (Coats et al., 1970;

Slater & Durrer, 1971; Thomas et al., 1972; Carter et al., 1974) could be attributed largely to the relative insensitivity of a reservoir simulator - i.e., a fairly wide range of reservoir description parameters will yield similar calculated performance data (Coats, 1969). Furthermore, according to Slater & Durrer (1971), a linear functional dependence appears to be particularly suitable when the reservoir possesses relatively high porosities and permeabilities. Note that, since the reservoir description parameters for each run, x_j^r, are multiplied in pairs (see Equation (4.2)), it appears that there is some cancellation of the error in solving for the a_{ij} coefficients.

Secondly, consider the statement of the linear functional relationship between the matching error (for any point of observation) and the reservoir description parameters as given in Equation (4.1):

$$\varepsilon = a_o + a_1 x_1 + a_2 x_2 + \ldots + a_J x_J \quad ,$$

where x_j $(j=1,2,\ldots,J)$ are the reservoir description parameters, the matching error, ε, is given by $\varepsilon = d^{obs} - d^{calc}$, and d^{obs} and d^{calc} are, respectively, the observed and calculated performance data at any point (which depend on the reservoir description parameters).

For convenience, let $\underline{x}_T$ and $\underline{x}$ represent the column vectors containing the true and estimated reservoir description parameters, respectively. Then the model on which Coats et al. base their algorithm is

$$\varepsilon = d^{obs}(\underline{x}_T,\underline{y},t) - d^{calc}(\underline{x}_T,\underline{y},t) = \underline{a}'\underline{x}_T + \delta \quad , \tag{4.5}$$

where $\underline{a}$ is the column vector of coefficients to be determined, δ is a random error, t is the independent time variable, and $\underline{y}$ represents all other independent and dependent reservoir parameters not included in $\underline{x}_T$ and $\underline{x}$. Rewriting Equation (4.5) yields

$$d^{obs}(\underline{x}_T,\underline{y},t) = d^{calc}(\underline{x}_T,\underline{y},t) + \underline{a}'\underline{x}_T + \delta \quad . \qquad (4.6)$$

Now, since an estimate of the reservoir description parameters, $\underline{x}$, must be used to replace the unknown true parameters, $\underline{x}_T$, the model given by Equations (4.5) becomes, in practice,

$$d^{obs}(\underline{x}_T,\underline{y},t) - d^{calc}(\underline{x},\underline{y},t) - \underline{a}'\underline{x} \quad , \qquad (4.7)$$

which, ideally, should be nearly zero. Substituting Equation (4.6) into (4.7) yields

$$d^{calc}(\underline{x}_T,\underline{y},t) + \underline{a}'\underline{x}_T + \underline{\delta} - d^{calc}(\underline{x},\underline{y},t) - \underline{a}'\underline{x} \quad ,$$

or, on rearranging terms,

$$d^{calc}(\underline{x}_T,\underline{y},t) - d^{calc}(\underline{x},\underline{y},t) + \underline{a}'(\underline{x}_T-\underline{x}) + \delta \quad . \qquad (4.8)$$

It is evident that this expression will, in general, and regardless of the value of the a_{ij} coefficients, approach zero only when $\underline{x} \approx \underline{x}_T$.

Because the algorithm of Coats et al. uses a reservoir simulator with several different sets of estimated description parameters, it is, of course, quite possible that the expression given by (4.8) yields a result close to zero. If the $\underline{x}^r$ are selected such that they are

centred about the true reservoir description parameters, $\underline{x}_T$, the error in estimating $\underline{x}_T$ can be balanced. Thus, if the bounds on the $\underline{x}^r$ are symmetrical about $\underline{x}_T$ (i.e., for each j, $x_{ju} - x_j = x_j - x_{jl}$), the random number generator will ensure a fairly even distribution of the $\underline{x}^r$ about $\underline{x}_T$, and the model presented in Equation (4.5) should be reasonable.

Experimental results support this argument. As noted above, the success of the method developed by Coats et al. is well-documented in numerous case studies; and these indicate that the algorithm also yields satisfactory results for multiphase flow as well as when alternative zonation patterns are employed.

In practical situations, it is not likely that bounds on the reservoir description parameters (estimated from available geological data) will be absolutely (or even approximately) symmetrical about the true description parameters. In that case, the algorithm of Coats et al. tends almost exclusively to choose as reservoir description estimates the extremal, bounding values. Results obtained by Dr. S.M. Farouq Ali and students at Pennsylvania State University (1973) evidence this characteristic problem. Working with a hypothetical, two-dimensional gas reservoir consisting of six zones, it was desired to determine the absolute permeabilities. The available performance data and initial reservoir data, as well as bounds on the unknown permeabilities, are listed in Table 1, and the zonation is shown in Figure 10. (Note the asymmetry of the imposed bounds about the true reservoir permeabilities.) Results of five different applications of the automatic history-matching algorithm, and the true reservoir permeabilities, are provided in Table 2.

TABLE 1

RESERVOIR DATA AND PERMEABILITY BOUNDS

Well Location: (2,2)

Production Rate: 0.3277 m^3/s

All blocks of equal size: $\Delta x = \Delta y = 267$ m

Thickness: 9.14 m

Temperature: 360°K

Porosity: 0.20

Initial Pressure: 3447.4 kPa

Time (days)	Pressure at Well (kPa)
100	3226.7
200	3062.0
300	2898.6
400	2734.5
500	2569.7
600	2402.8
700	2233.9
800	2063.6
900	1889.9
1000	1714.0
1100	1534.1

Permeability Bounds (μm^2)

$0.049 < k_1 < 0.296$

$0.987 < k_2 < 4.935$

$0.099 < k_3 < 0.493$

$0.197 < k_4 < 0.296$

$0.987 < k_5 < 5.922$

$0.010 < k_6 < 9.869$

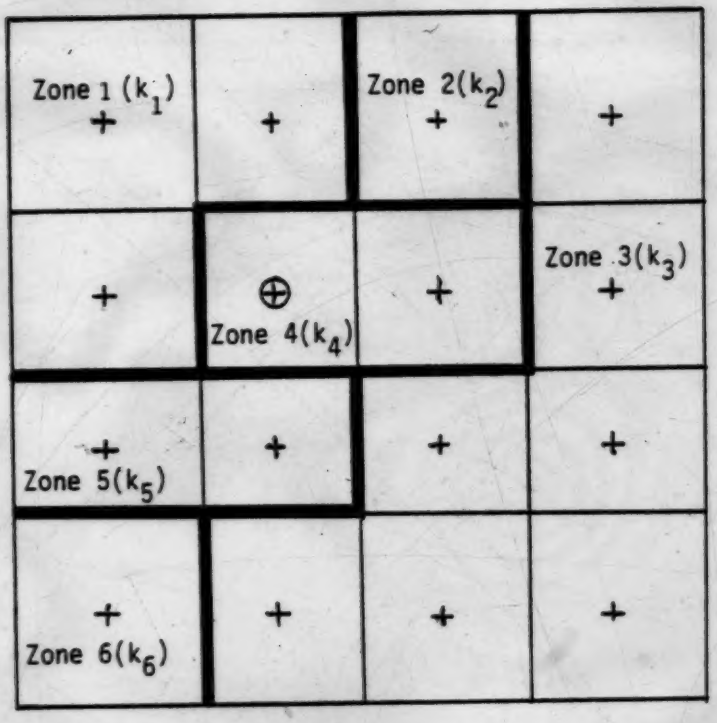


Figure 10. Reservoir Zonation

TABLE 2

ESTIMATED PERMEABILITIES (μm^2)

Run	#1	#2	#3	#4	#5
Zone 1	0.296*	0.049*	0.049*	0.049*	0.049*
Zone 2	2.061	1.433	0.987*	4.935*	0.987*
Zone 3	0.493*	0.493*	0.099*	0.493*	0.099*
Zone 4	0.197*	0.197*	0.197*	0.296*	0.197*
Zone 5	0.987*	0.987*	0.987*	0.987*	1.135
Zone 6	3.878	0.010*	0.010*	0.010*	0.010*

* indicates an extremal value

TRUE PERMEABILITIES (μm^2)

Zone 1	0.099
Zone 2	1.974
Zone 3	0.099
Zone 4	0.148
Zone 5	2.467
Zone 6	0.0001

Further analysis of the problem of asymmetrical bounds was carried out using the computer programs developed in Chapter 3 and in earlier sections of Chapter 4. Considering a hypothetical, two-dimensional, three-phase petroleum reservoir consisting of six zones, four applications of the algorithm of Coats et al. were used to estimate the absolute permeabilities. The first used bounds symmetrical about the true reservoir permeability; the second employed some asymmetrical bounds; the third used asymmetrical bounds entirely; and the fourth used intervals not containing the true permeabilities. The available performance and initial reservoir data and bounds on the unknown permeabilties are summarized in Table 3, and the zonation is shown in Figure 11. Results of the study are set out in Table 4. One observes that, as stated earlier, the history-matching algorithm selects extreme values the more frequently the less symmetrical the bounds on the permeabilities become about the true reservoir permeabilities.

Coats et al. noted that, even in the case of symmetrical bounds (which they and all others exclusively employed in tests), the algorithm often chooses at least one or two reservoir description parameters at their upper or lower limits. They therefore suggested that repeated passes could, if desired, be made with shifted limits on the necessary parameters until all estimated values lie within specified bounds. However, they point out that it is far more convenient to choose bounds representing a reasonable range, apply the algorithm once, and accept any resulting extremal values selected.

TABLE 3

RESERVOIR DATA AND PERMEABILITY BOUNDS

Relative Permeabilities and Capillary Bounds

Oil-Water				Oil-Gas			
S_w	k_{rw}	k_{row}	P_{cow}(Pa)	S_{o_2}	k_{rg}	k_{rog}	P_{cgo}(Pa)
0.1	0	1	28344	0.1	0.520	1	31144
0.2	0.0016	0.875	655	0.2	0.410	0.009	462
0.3	0.081	0.735	496	0.3	0.310	0.031	290
0.4	0.0259	0.590	421	0.4	0.220	0.062	138
0.5	0.0672	0.420	352	0.5	0.140	0.110	-7
0.6	0.1000	0.210	283	0.6	0.080	0.190	-152
0.7	0.1400	0.070	214	0.7	0.030	0.335	-297
0.8	0.2000	0.016	145	0.8	0.005	0.570	-441
0.86	0.2500	0	76	0.89	0	1	-586

where k_{row} is the relative permeability of oil in the oil-water system, and k_{rog} is the relative permeability of oil in the oil-gas system; and $S_g = 1 - S_{wir} - S_{o_2}$, where S_{o_2} is the oil saturation in the two-phase gas-oil system, with irreducible water present.

Formation Volume Factors, Fluid Viscosities, Gas Solubility

p(kPa)	B_o(m^3/sm^3)	B_w(m^3/sm^3)	B_g(m^3/sm^3)	μ_o(mPa.s)	R_{sw}(sm^3/sm^3)	R_{so}(sm^3/sm^3)
0	1.0000	1.0000	1.0000	3.700	0	0
1379	1.1160	1.0000	0.0845	3.050	2	13.53
2758	1.1245		0.0413	2.879	2	19.77
4137	1.1375		0.0270	2.622	2	25.11
5516	1.1500		0.0198	2.470	2	30.10
6853	1.1623	0.99354	0.0156	2.390	2	34.91
11032	1.1569		0.0091	2.545	2	34.91
15168	1.1569		0.0065	2.700	2	34.91

TABLE 3 (continued)

μ_g = 0.0167 mPa•s; μ_w = 0.43 mPa•s
$c_r = 0.5\times10^{-6}$ kPa^{-1}
Initial porosity: 0.09
Initial pressure: 3000 kPa
Initial oil saturation: 0.70
Initial water saturation: 0.30
$\rho_{o,sc}$ = 806 kg/m^3 ; $\rho_{w,sc}$ = 1109 kg/m^3 ; $\rho_{g,sc}$ = 1.2815 kg/m^3
Well locations: (1,4); (3,1); (3,3); (3,5); (5,2); (5,6); (6,4)
Oil production rate in each well: 9.2×10^{-5} m^3/s
Thickness: 12 m
Depth of block (1,4): 2060 m
Dip: 7° downward in positive x-direction
Bubble point pressure: 6853 kPa

Dimensions (m)

Δx_1 = 1500	Δy_1 = 500
Δx_2 = 1200	Δy_2 = 200
Δx_3 = 2000	Δy_3 = 500
Δx_4 = 500	Δy_4 = 100
Δx_5 = 500	Δy_5 = 1500
Δx_6 = 200	Δy_6 = 100

Performance Data: 272 days; N = 12 runs

Well Block	Pressure (kPa)
(1,4)	2822
(3,1)	2975
(3,3)	2991
(3,5)	3007
(5,2)	3169
(5,6)	3133
(6,4)	3408

TABLE 3 (Continued)

Permeability Bounds (μm^2)

Case 1 (symmetrical)

$0.3 < k_1 < 0.9$

$0.6 < k_2 < 1.0$

$0.1 < k_3 < 0.7$

$0.3 < k_4 < 1.5$

$0.2 < k_5 < 1.4$

$0.1 < k_6 < 0.9$

Case 2 (partially symmetrical)

$0.3 < k_1 < 0.9$

$0.7 < k_2 < 1.2$

$0.1 < k_3 < 0.7$

$0.2 < k_4 < 1.1$

$0.2 < k_5 < 1.4$

$0.4 < k_6 < 0.9$

Case 3 (asymmetrical)

$0.2 < k_1 < 0.7$

$0.7 < k_2 < 1.2$

$0.35 < k_3 < 0.7$

$0.2 < k_4 < 1.1$

$0.1 < k_5 < 0.9$

$0.4 < k_6 < 0.9$

Case 4 (non-inclusive)

$0.7 < k_1 < 1.5$

$0.2 < k_2 < 0.6$

$0.8 < k_3 < 1.0$

$0.2 < k_4 < 0.8$

$1.0 < k_5 < 1.4$

$0.6 < k_6 < 0.9$

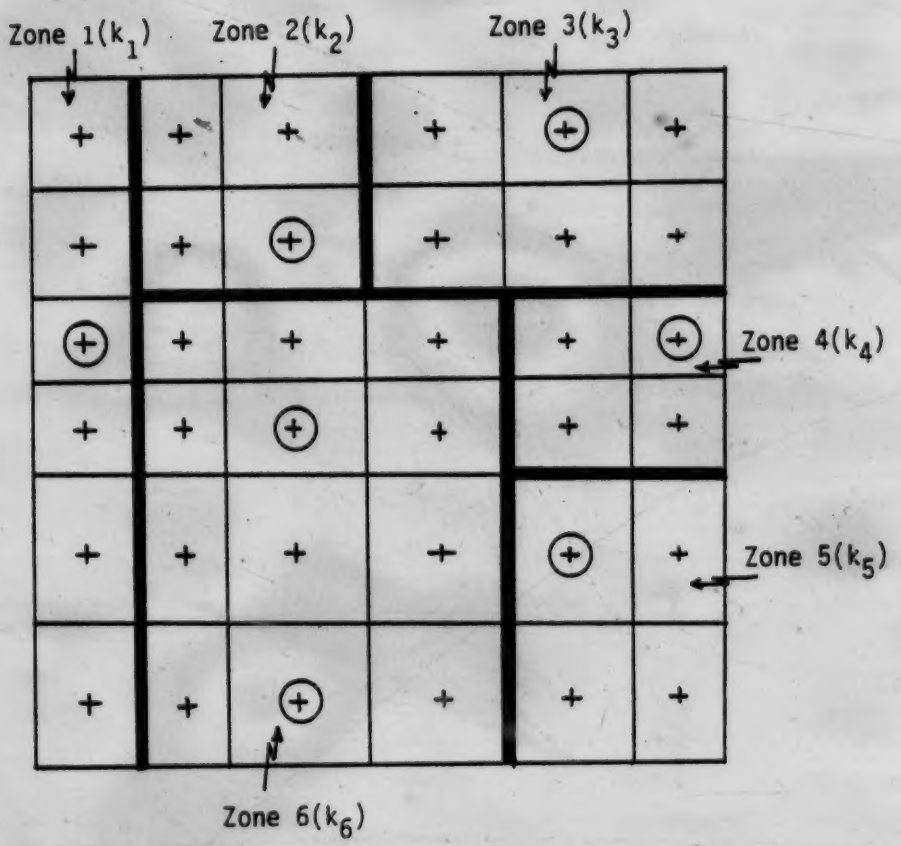


Figure 11. Reservoir Zonation

TABLE 4

ESTIMATED AND TRUE PERMEABILITIES (μm^2)

Case	#1	#2	#3	#4	True
Zone 1	0.620	0.651	0.7*	0.7*	0.6
Zone 2	1.0*	0.7*	0.7*	0.6*	0.8
Zone 3	0.376	0.7*	0.35*	0.8*	0.4
Zone 4	0.921	1.1*	1.1*	0.8*	0.9
Zone 5	0.819	0.711	0.9*	1.0*	0.8
Zone 6	0.483	0.779	0.4*	0.6*	0.5

* indicates an extremal value

In practice, it seems therefore appropriate to use the automatic history-matching algorithm of Coats et al. (1970) for single-phase flow, and multiphase flow where relative permeability curves are well defined. If it results in selection of most of the extremal values, a readjustment of the bounding values should produce reasonable data on a supplementary pass. The relative insensitivity of a reservoir simulator to small changes in permeability appears to enhance the acceptability of the results obtained by use of this algorithm. The reason for this is that small variations in parameter estimates will have little effect on future performance predictions made by the simulator.

CHAPTER FIVE

A DIRECT APPROACH TO AUTOMATIC HISTORY-MATCHING

In 1972, Veatch & Thomas published an entirely new approach to automatic history-matching. While previous (and subsequent) methods for automatic history-matching resemble the algorithm of Coats et al. (1970), in that they are essentially ex post facto techniques (i.e., after numerous simulations using different initial data, various optimization and/or statistical methods based on comparisons of the known performance history with the results of the simulations are employed to determine acceptable estimates of the reservoir description parameters), Veatch & Thomas developed an easily implemented direct method for inverse reservoir simulation that substantially reduces the demand for computer time.

By treating the finite difference analogues of the partial differential equations which model multiphase flow as a system of equations in the unknown reservoir description parameters, and employing performance histories over several periods of time as input data, it is possible to determine directly the description parameters in a single computer run. The particular advantage of this method is that it does not require implementation of a reservoir simulator, and thus eliminates computational inaccuracies arising from inherent inadequacies of the simulator. It should also be observed that the equations are solved directly, and that the solution method is therefore applicable to multiphase, compressible flow in heterogeneous reservoirs. Additionally, Veatch & Thomas included provisions for constructing 'best' estimates of reservoir description parameters in cases where no unique solutions can be computed.

Veatch & Thomas' direct method for automatic history-matching represents a considerable saving in time and effort in handling such problems, and appears to have the potential to produce superior estimates of reservoir description parameters. Since, as indicated in the literature review, this procedure has so far attracted little comment, it is a matter of some interest to analyze it here as an alternative to Coats et al.'s algorithm.

Derivation of the Algorithm

Using the standard technique for representing a petroleum reservoir as a grid-block system, any partial differential phase equation required for describing the flow mechanics in any block (i,j) in a two-dimensional areal reservoir may be written in the form

$$(X^n k_x)_{i+1/2,j} - (X^n k_x)_{i-1/2,j} + (Y_n k_y)_{i,j+1/2} - (Y^n k_y)_{i,j-1/2} - (P^n \phi)_{i,j} = -q^n_{i,j} \quad , \tag{5.1}$$

where the permeabilities and porosity are unknown, and the X^n, Y^n and P^n are coefficients containing relative permeabilities, formation volume factors, fluid viscosities, spatial increments, and potentials, all at time level n (cf., for the case of three-phase flow, the equations derived in Chapter 3.) These coefficients, as well as the q^n terms, depend on both the pressure and saturation distributions throughout the grid-block system at several points in time; and since the only firm sources of information lie in blocks containing production or injection wells, various interpolation schemes must be employed to

estimate the required pressure and saturation distributions in the remainder of the reservoir.

Theoretically, a solution to any one phase equation should satisfy all other phase equations, and Veatch & Thomas consequently based their algorithm on a single, arbitrarily chosen phase equation. The use of Equation (5.1) for any phase, when combined with known reservoir pressure and saturation distributions at several points in time (which must be at least as numerous as one plus the number of unknowns in each block) thus results in a linear system of equations in the unknown reservoir description parameters.

Veatch & Thomas offered a detailed discussion of the matrix form of the resulting linear system of equations (which is of the familiar $\underline{A}\underline{x} = \underline{b}$ form), and suggested solving it in a least-squares sense by applying Householder transformations (see Householder, 1965). They observed that singular, nearly-singular, and ill-conditioned systems of equations led to non-unique solutions, and identified such cases from analysis of the diagonal elements of the matrix of coefficients.

Moreover, Veatch & Thomas chose to assign permeability values to the faces, rather than the centres, of each grid block (i.e., k_x was to be defined at $(i+1/2,j)$ and $(i-1/2,j)$ rather than at (i,j)), and the history-matching problem consequently contained five, rather than two, unknowns in any block. Given the large number of unknowns over an entire reservoir grid system, they suggested that only small groups of blocks be treated initially in order to construct local solutions. And since the most accurate pressure and saturation information on the reservoir lies in blocks containing production or injection wells, they further recommended that the initial computations be focussed about the well blocks.

Thus, Veatch & Thomas' history-matching algorithm begins by making an initial pass over the entire reservoir in order to establish the systems of equations about each well block. Each system of equations is then tested for uniqueness; and if a unique solution can be computed, the reservoir description parameters are calculated. The systems of equations associated with the periphery of each block for which a unique solution was found are thereafter determined. If these systems produce unique solutions, the reservoir description parameters are computed, and the areas are gradually and uniformly enlarged over the reservoir until all possible unique solutions are obtained. If no unique solution can be found, the algorithm begins a non-unique construction of the solution. The computations in that case resemble the previous automatic history-matching algorithms, in that they require the imposition of upper and lower bounds on the reservoir description parameters, and are based on least-squares techniques in which the error in estimation is to some degree minimized.

In tests of their algorithm on several hypothetical reservoirs, Veatch & Thomas found generally excellent agreement between actual and computed reservoir description parameters in the case of incompressible flow, for which unique solutions could be obtained. But when non-unique solutions were encountered, they conceded that "there is no assurance that the results of the prediction phase will be reliable", and, "as the prediction is made farther into the future, the results become even worse." The one example of compressible flow treated by Veatch & Thomas led to a non-unique solution, and implementation of a bounded least-squares technique produced reasonable results. However, as in the case of the automatic history-matching algorithm of Coats et al. (1970) the

apparent success of the history-match was due to imposition of symmetrical bounds about the true values of the reservoir description parameters.

Discussion

While Veatch & Thomas' method for automatic history-matching is potentially useful, it contains several inherent weaknesses. Its main disadvantage is its tendency to be very sensitive to the input pressure distributions. (Such sensitivity to minor variations in reservoir pressure was observed as early as 1964 by Jacquard.) This is illustrated by a simplified version of the direct method applied to a hypothetical reservoir.

Consider the oil phase equation appropriate for modelling three-phase flow (Equation (3.7a)):

$$\frac{\partial}{\partial x}\left(\frac{A_x k_x k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial x}\right)\Delta x + \frac{\partial}{\partial y}\left(\frac{A_y k_y k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial y}\right)\Delta y + \frac{\partial}{\partial x}\left(\frac{A_z k_z k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial z}\right)\Delta z + q_o^* = V_b \frac{\partial}{\partial t}\left(\phi \frac{S_o}{B_o}\right) . \tag{5.2}$$

For convenience, suppose the reservoir is two-dimensional, and $k_x = k_y = k$ throughout the reservoir. Then, for any given block (i,j), Equation (5.2) may be written by finite difference techniques as

$$k_{i,j}\left\{\left(\frac{k_{ro}}{\mu_o B_o}\right)^n_{i+1/2,j}\frac{A_x}{\Delta x}\left(\Phi^{n+1}_{i+1,j}-\Phi^{n+1}_{i,j}\right) - \left(\frac{k_{ro}}{\mu_o B_o}\right)^n_{i-1/2,j}\frac{A_x}{\Delta x}\left(\Phi^{n+1}_{i,j}-\Phi^{n+1}_{i-1,j}\right) + \left(\frac{k_{ro}}{\mu_o B_o}\right)^n_{i,j+1/2}\frac{A_y}{\Delta y}\left(\Phi^{n+1}_{i,j+1}-\Phi^{n+1}_{i,j}\right) - \left(\frac{k_{ro}}{\mu_o B_o}\right)^n_{i,j-1/2}\frac{A_y}{\Delta y}\left(\Phi^{n+1}_{i,j}-\Phi^{n+1}_{i,j-1}\right)\right\}$$

$$= V_b \frac{1}{\Delta t}\left\{\left(\phi \frac{S_o}{B_o}\right)^{n+1}_{i,j} - \left(\phi \frac{S_o}{B_o}\right)^{n}_{i,j}\right\} - q^{*}_{o_{i,j}} \quad , \qquad (5.3)$$

where the various terms are as defined in Chapter 3.

To simplify the problem further, assume that the porosity is known, so that Equation (5.3) contains the permeability as the only unknown. Knowledge of pressure and saturation distributions at two points in time will then suffice for estimating the permeability in each grid-block throughout the reservoir.

A three-phase, two-dimensional areal reservoir, divided into a 7×7 grid-block system, was employed in the sample calculations. All necessary input data, including the true reservoir permeabilities and the pressure and saturation distributions at two time points, are shown in Table 5. To test the sensitivity of Equation (5.3) to the pressure distribution, the pressures at the first time step were perturbed in various blocks by increments as small as 6.895 kPa (1 psi). When the pressure values in each block (i,j), i = 1,2,3; j = 1,2,3,4 were increased by 6.895 kPa, the resulting permeability estimates in those and surrounding blocks were strongly affected, with some estimates displaying shifts in the range of 0.7 μm^2 to 1.8 μm^2. In fact, the solution generated several negative estimates of permeability.

It is possible to provide an heuristic argument to demonstrate the sensitivity of Equation (5.3) to pressure distributions. Consider the coefficient on the left hand side of Equation (5.3):

$$\left(\frac{k_{ro}}{\mu_o B_o}\right)^{n}_{i+1/2,j} \frac{A_x}{\Delta x}\left(\Phi^{n+1}_{i+1,j} - \Phi^{n+1}_{i,j}\right) - \left(\frac{k_{ro}}{\mu_o B_o}\right)^{n}_{i-1/2,j} \frac{A_x}{\Delta x}\left(\Phi^{n+1}_{i,j} - \Phi^{n+1}_{i-1,j}\right)$$

$$+ \left(\frac{k_{ro}}{\mu_o B_o}\right)^{n}_{i,j+1/2} \frac{A_y}{\Delta y}\left(\Phi^{n+1}_{i,j+1} - \Phi^{n+1}_{i,j}\right) - \left(\frac{k_{ro}}{\mu_o B_o}\right)^{n}_{i,j-1/2} \frac{A_y}{\Delta y}\left(\Phi^{n+1}_{i,j} - \Phi^{n+1}_{i,j-1}\right) \quad .$$

TABLE 5

RESERVOIR DATA

Well locations: (1,7); (4,1); (4,4); (7,7)

Oil production rates: 9.2×10^{-5} m^3/s in blocks (1,7),(4,1),(7,7)

1.9×10^{-4} m^3/s in block (4,4)

Excluded blocks: (1,1); (7,1)

All blocks of equal size: $\Delta x = \Delta y = 30.5$ m

Thickness: 12.2 m

Porosity: 0.2

Depth below sea level: 305 m

Initial oil pressure: 13100 kPa

Permeability: 0.296 μm^2

Initial oil saturation: 0.550

Initial gas saturation: 0.050

Oil Pressure Distribution (60 days)

11356	11383	11397	11397	11397	11383	11356
11383	11390	11397	11397	11397	11390	11383
11397	11397	11397	11390	11397	11397	11397
11404	11404	11390	11363	11390	11404	11404
11411	11411	11397	11390	11397	11411	11411
11418	11411	11404	11397	11404	11411	11418
0	11411	11397	11383	11397	11411	0

Oil Saturation Distribution (60 days)

0.507	0.535	0.510	0.509	0.510	0.535	0.507
0.518	0.501	0.509	0.515	0.509	0.501	0.518
0.514	0.508	0.501	0.524	0.501	0.508	0.514
0.509	0.511	0.559	0.523	0.559	0.511	0.509
0.512	0.511	0.500	0.525	0.500	0.511	0.512
0.513	0.510	0.502	0.522	0.502	0.510	0.513
0	0.509	0.521	0.508	0.521	0.509	0

TABLE 5 (Continued)

Water Saturation Distribution (60 Days)

0.401	0.402	0.401	0.401	0.401	0.402	0.401
0.402	0.401	0.401	0.401	0.401	0.401	0.402
0.401	0.401	0.401	0.402	0.401	0.401	0.401
0.401	0.401	0.403	0.399	0.403	0.401	0.401
0.401	0.401	0.401	0.402	0.401	0.401	0.401
0.401	0.401	0.401	0.402	0.401	0.401	0.401
0	0.401	0.402	0.401	0.402	0.401	0

Oil Pressure Distribution (100 days)

10384	10411	10425	10425	10425	10411	10384
10411	10418	10425	10425	10425	10418	10411
10425	10425	10425	10418	10425	10425	10425
10439	10432	10418	10390	10418	10432	10439
10439	10439	10432	10418	10432	10439	10439
10446	10439	10432	10425	10432	10439	10446
0	10439	10425	10404	10425	10439	0

Oil Saturation Distribution (100 days)

0.495	0.502	0.487	0.486	0.487	0.502	0.495
0.492	0.480	0.487	0.492	0.487	0.480	0.492
0.492	0.487	0.479	0.494	0.479	0.487	0.492
0.488	0.488	0.516	0.501	0.516	0.488	0.488
0.491	0.489	0.480	0.493	0.480	0.489	0.491
0.492	0.488	0.481	0.492	0.481	0.488	0.492
0	0.486	0.494	0.492	0.494	0.486	0

In those parts of the reservoir in which the pressure distribution tends to be uniform, this expression will approximate zero. Since each $k_{i,j}$ is determined by dividing this value into the right hand side constant of Equation (5.3), it is evident that small changes in the value could have a significant effect on the outcome of the history-match. This point was noted by Veatch & Thomas: "The greatest deviations occurred in the farthest regions from the wells where the pressure distributions from one time to the next are not significantly perturbed. As a consequence we begin to see the effects of ill-conditioning where small errors in the coefficient matrix are greatly magnified in the solution vector."

The sensitivity of this direct history-matching method to the pressure distributions was further corroborated during personal communication with Dr. G.W. Thomas.

Although the direct method for automatic history-matching is advantageous in that it performs independently of a reservoir simulator, its sensitivity appears to render it ineffective as a viable approach to automatic history-matching in practical situations, where utilization of various interpolation schemes can lead to widely different estimated pressure distributions.

CHAPTER SIX

SOME MATHEMATICAL CONSIDERATIONS

Theoretical Suitability of Darcy's Equation of Flow

As discussed in Chapter 3, mathematical models simulating multiphase, multi-dimensional fluid flow in a petroleum reservoir are ultimately based on Darcy's equation of flow (used in conjunction with the law of conservation of mass). The effectiveness of this equation as the fundamental relationship governing fluid flow behaviour in porous media has been established by experimental testing, and a large body of supporting material (applied to actual field situations) is readily available (see, for example, Muskat, 1949; Collins, 1961; Scheidegger, 1974; or Thomas, 1982). But equally important for the significance of the mathematical model is that Darcy's equation of flow has some theoretical foundations.

In 1975, Houpeurt described a method - first published in 1955 in Revue de l'Institut Français du Pétrole - by which Darcy's equation of flow could be deduced from theoretical considerations of the Navier-Stokes equations of motion.

In Cartesian coordinates, the Navier-Stokes equations of motion for an incompressible fluid with constant viscosity may be written as

$$\rho \frac{Du}{Dt} = \frac{-\partial p}{\partial x} + F_x + \mu \nabla^2 u \quad ,$$

$$\rho \frac{Dv}{Dt} = \frac{-\partial p}{\partial y} + F_y + \mu \nabla^2 v \quad ,$$

and

$$\rho \frac{Dw}{Dt} = \frac{-\partial p}{\partial z} + F_z + \mu \nabla^2 w \quad ,$$

where u, v, and w are the velocities in the x, y, and z directions, respectively, $\vec{F}$ is a vector of body force density, and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \quad .$$

The law of conservation of mass (or the equation of continuity) is, for this case,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad .$$

For steady-state fluid flow,

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0 \quad .$$

Furthermore, assuming flow to be isothermal, laminar, viscous, and one-dimensional along the x-axis, the equation of continuity in the absence of external body forces is simply

$$\frac{\partial u}{\partial x} = 0 \quad ,$$

and the Navier-Stokes equations reduce to

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\mu}{\rho} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad ,$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \ ,$$

and

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \ ,$$

which may be combined and written as

$$\frac{dp}{dx} = \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \ . \qquad (6.1)$$

Houpeurt then dealt with two particular cases of flow. In the first instance, he considered flow along the x-direction through a three dimensional semi-infinite strip, defined by $0 < x < L$, $-\infty < y < \infty$, $-e/2 < z < e/2$, so that Equation (6.1) could be written as

$$\frac{dp}{dx} = \mu\frac{d^2 u}{dz^2} \ . \qquad (6.2)$$

Since the two terms in Equation (6.2) depend upon different variables, each term may be set equal to a constant and integrated. For the boundary conditions

$$x = 0 \ , \ p = p_1$$
$$x = L \ , \ p = p_2 \qquad (p_2 < p_1) \ ,$$

and

$$u = 0,\ z = \pm \frac{e}{2} \quad ,$$

integration of each term of Equation (6.2) yields

$$p = p_1 - \frac{p_1 - p_2}{L} x \quad ,$$

and

$$u = \frac{1}{2\mu} \frac{p_1 - p_2}{L} \left(\frac{e^2}{4} - z^2\right) \quad .$$

Calculation of the output of fluid per unit time, q, over a distance b along the strip yields

$$q = 2\int_0^{e/2} b u dz$$

$$= be \frac{e^2}{12\mu} \frac{p_1 - p_2}{L} \quad ,$$

i.e., this is the fluid output per unit time through the cross-sectional area be. For a porous medium with porosity ϕ, the total output of fluid per unit time across the strip, Q, is then given by

$$Q = A \frac{\phi e^2}{12} \frac{1}{\mu} \frac{p_1 - p_2}{L} \quad , \qquad (6.3)$$

where A is the total cross-sectional area along the strip.

In the second instance, Houpeurt considered flow through a cylindrical porous medium, and, by methods similar to those used above

derived

$$Q = A \frac{\phi r^2}{8} \frac{1}{\mu} \frac{p_1 - p_2}{L} \quad , \qquad (6.4)$$

where r is the radius of the cylinder.

Comparison of Equations (6.3) and (6.4) then led Houpeurt to suggest that, for any cross-sectional area A of a porous medium through which the direction of flow is normal, the effect of a pressure gradient on the output of fluid per unit time, q, would be of the form

$$q = A \frac{k}{\mu} \frac{dp}{dx} \quad ,$$

where k is a constant. This is, in fact, the result obtained experimentally by Darcy in 1856.

The preceding arguments allow the observation that the problem of obtaining a fundamental expression for fluid flow in a porous medium reduces, essentially, to determining the macroscopic behaviour of a porous medium which exhibits microscopic heterogeneity. In other words, since the problem lies in the existence of two widely differing length scales, or 'scales of variation' (the smaller relating to heterogeneity, and the larger to macroscopic behaviour), the objective must be a characterization of the large scale behaviour of the medium by eliminating small scale variation from the known equations which describe flow behaviour on a small scale (i.e., the Navier-Stokes equations of motion).

In recent years, several systematic, rigorous methods for dealing with such problems have been developed. These are based on various averaging techniques, including spatial, temporal, and stochastic

averaging. In 1977, Keller developed the so-called 'two-space method', or 'method of multiple scales', as means for deriving simplified equations. The method can handle large amplitude variations in the coefficients which may occur over a small length scale, and Keller clearly illustrated the technique by applying it to a boundary value problem for a second order partial differential equation.

In a 1980 paper, Keller further illustrated the use of the two-space method for the simple case of heat conduction in one dimension, and then used this approach to derive Darcy's equation of flow from the Navier-Stokes equations. The following is a brief description of the two-space method as applied by Keller to flow through a porous medium.

Consider an equation with coefficients that vary on a small spatial scale. These coefficients are first represented by writing them as functions of x/ε, where x is a position vector and ε a small parameter proportional to the length scale. That this will cause a function $f(x/\varepsilon)$ to be subject to rapid variation is evident by noting that its derivative with respect to x, i.e.

$$\frac{\partial}{\partial x} f\left(\frac{x}{\varepsilon}\right) = \frac{1}{\varepsilon} f'\left(\frac{x}{\varepsilon}\right) \quad ,$$

is large for small values of ε, even though f' may be bounded.

For a flow of a compressible, viscous fluid through a porous medium, let ε denote the ratio of the pore diameter to the macroscopic scale. The relationships governing flow behaviour on the microscopic scale (i.e., in the interior of the pores) are then the Navier-Stokes equations of motion, the equation of continuity, and the equation of state, namely:

$$\rho\left(\frac{\partial u}{\partial t} + u\cdot\nabla u\right) = -\nabla p + \mu\left(\nabla^2 + \frac{1}{3}\nabla\nabla\cdot\right)u + f, \qquad (6.5)$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho u) = 0 \quad , \qquad (6.6)$$

and

$$\rho = \rho(p) \quad , \qquad (6.7)$$

where ρ, p, u, and μ are, respectively, the fluid density, pressure, velocity, and viscosity coefficient, and f is the external body force per unit volume. In addition, it is assumed that $u = 0$ on the surface of the pores, to account for viscosity effects.

Keller first introduced the variables y and τ, defined by

$$y = x/\varepsilon \quad ,$$
$$\tau = t/\varepsilon^{1/2} \quad ,$$

and wrote u, μ, ρ, p, and f in the forms

$$u = \varepsilon^{1/2}\tilde{u}(x,y,\tau,\varepsilon) \quad ,$$

$$\mu = \varepsilon^{3/2}\tilde{\mu} \quad ,$$

$$\rho = \tilde{\rho}(x,y,\tau,\varepsilon) \quad ,$$

$$p = \tilde{p}(x,y,\tau,\varepsilon) \quad ,$$

and

$$f = \tilde{f}(x,y,\tau,\varepsilon) \quad .$$

Note that μ is small of order $\varepsilon^{3/2}$, i.e., the viscosity decreases proportionally to the decrease in pore size. Also, the powers of ε employed vary among application of the two-space methods to different problems. There is no formula to determine appropriate powers, and intuition or trial-and-error must be used to choose powers that provide acceptable final results. (For further references to this problem, see, e.g., Bensoussan et al. (1978), Chap. 2; Kogelman & Keller (1973); or Larsen (1975).) Replacing ∇ by $\nabla_x + \frac{1}{\varepsilon}\nabla_y$, and substituting the above variables into Equations (6.5)-(6.7) yields

$$\tilde{\rho}[\frac{\partial \tilde{u}}{\partial \tau} + \tilde{u}\cdot(\nabla_y+\varepsilon\nabla_x)\tilde{u}] = -(\frac{1}{\varepsilon}\nabla_y+\nabla_x)\tilde{p}$$

$$+ \tilde{\mu}[(\nabla_y+\varepsilon\nabla_x)^2 + \frac{1}{3}(\nabla_y+\varepsilon\nabla_x)(\nabla_y+\varepsilon\nabla_x)\cdot]\tilde{u} + \tilde{f} \quad , \qquad (6.8)$$

$$\frac{\partial \tilde{\rho}}{\partial \tau} + (\nabla_y+\varepsilon\nabla_x)\cdot(\tilde{\rho}\tilde{u}) = 0 \quad , \qquad (6.9)$$

and

$$\tilde{\rho} = \rho(\tilde{p}) \quad . \qquad (6.10)$$

Assuming that $\tilde{u}$, $\tilde{\rho}$, $\tilde{p}$, and $\tilde{f}$ have a regular dependence upon ε, they can be expanded into

$$\tilde{u}(x,y,\tau,\varepsilon) = u_0(x,y,\tau) + \varepsilon u_1(x,y,\tau) + o(\varepsilon) , \qquad (6.11)$$

$$\tilde{\rho} = \rho_0 + \varepsilon\rho_1 + o(\varepsilon) , \qquad (6.12)$$

$$\tilde{p} = p_0 + \varepsilon p_1 + o(\varepsilon) , \qquad (6.13)$$

and

$$\tilde{f} = f_0 + \varepsilon f_1 + o(\varepsilon) . \qquad (6.14)$$

Substituting (6.11)-(6.14) into Equations (6.8)-(6.10), and equating coefficients of the lowest power of ε in each equation yields

$$\nabla_y p_0(x,y,\tau) = 0,$$

$$\frac{\partial \rho_0}{\partial \tau} + \nabla_y \cdot (\rho_0 u_0) = 0 , \qquad (6.15)$$

and

$$\rho_0 = \rho(p_0) .$$

Similarly, equating coefficients for the next lowest power of ε gives

$$\rho_0\left(\frac{\partial u_0}{\partial \tau} + u_0 \cdot \nabla_y u_0\right) = -\nabla_y p_1 + \tilde{\mu}\left(\nabla_y^2 + \frac{1}{3}\nabla_y \nabla_y \cdot\right) u_0$$

$$+ f_0 - \nabla_x p_0 , \qquad (6.16)$$

$$\frac{\partial \rho_1}{\partial \tau} + \nabla_y \cdot (\rho_0 u_1 + \rho_1 u_0) + \nabla_x \cdot (\rho_0 u_0) = 0 ,$$

and

$$\rho_1 = \rho_p(p_0)p_1 \ .$$

From here, Keller proceeded to determine the variables upon which each function depends, and after averaging (i.e., integrating over a large domain, D, of the fluid, dividing the integral by the volume V of D, and letting D and V tend to infinity), obtained two relations which generalize Darcy's law for nonlinear, time-dependent, compressible flows:

$$\tilde{u}_0(x,\tau) = \overline{U}[x,\tau,\rho_0,f_0 - \nabla_x p_0]$$

$$= \lim_{V\to\infty} \frac{1}{V} \int_D U[x,y,\tau,\rho_0,f_0 - \nabla_x p_0]dy \quad ,$$

and

$$\overline{p}_1(x,\tau) = \overline{P}[x,\tau,\rho_0,f_0 - \nabla_x p_0]$$

$$= \lim_{V\to\infty} \frac{1}{V} \int_D P[x,y,\tau,\rho_0,f_0 - \nabla_x p_0]dy \quad ,$$

where the solutions to (6.15) and (6.16) are written as functionals of ρ_0 and $f_0 - \nabla_x p_0$ in the form

$$u_0(x,y,\tau) = U[x,y,\tau,\rho_0,f_0-\nabla_x p_0] \ ,$$

and

$$p_1(x,y,\tau) = P(x,y,\tau,\rho_0,f_0-\nabla_x p_0] .$$

With these relationships in mind, Keller then considered several special cases which more closely resemble the usual form of Darcy's equation - flow of incompressible fluid, steady incompressible flow with constant viscosity, and steady compressible flow. In the case of steady, incompressible flow, the simplified equation is the Darcy equation of flow in one of its common forms:

$$\overline{u}_0(x) = \overline{A}(x)[f_0(x) - \nabla_x p_0(x)] ,$$

where $\overline{A}(x)$ is the cross-sectional area normal to the direction of flow.

Thus, based on the equations governing flow on the microscopic level, it is possible to obtain, from purely theoretical considerations, a characterization of large scale flow behaviour in a porous medium.

Alternative Approaches to the Inverse Problem

As shown in Chapter 2, petroleum engineers have devoted considerable efforts to the development of feasible automatic methods that solve the inverse problem. But analogous inverse problems also occur frequently in science, medicine, and engineering, and a large body of knowledge exists elsewhere in the literature (see, for example, Kagiwada, 1974; Carasso & Stone, 1975; or references provided by Sagar et al., 1975. More extensive reference listings may be found in Payne (1975) and Tikhonov & Arsenin (1977).)

A somewhat less complex version of the problem of parameter identification in an underground reservoir arises when considering groundwater flow. The equation modelling such flow is very similar to that which models single-phase flow in a petroleum reservoir; and numerous papers seeking to determine the spatially varying coefficients of permeabilty and porosity are to be found in journals of water resources and hydrology. It is interesting to note that most of these attempts follow direct approaches, rather than the ex post facto methods almost exclusively used by petroleum engineers.

Proposed methods for solving the inverse problem (often for the steady-state condition only) include: linear programming (Kleinecke, 1971), automatic solutions (Emsellem & de Marsily, 1971), use of subjective information (Lovell et al., 1972; Nutbrown, 1975), Galerkin solutions (Frind & Pinder, 1973), finite elements (Neuman, 1973), direct identification based on approximated derivatives of the dependent variable (Sagar et al., 1975; Yakowitz & Noren, 1976), and quadratic programming (Chang & Yeh, 1976). However, there is little discussion of the sensitivity of calculated solutions to the required input data and few practical case studies have been reported.

A common feature of these direct algorithms is that they demand prior knowledge of the derivatives of the dependent variable (i.e., the pressure) over at least a portion of the reservoir. In two papers published in 1981, Richter studied the inverse problem for underground reservoirs using such input data, and analyzed possible solutions and solution sensitivity under various conditions.

In one paper, Richter (1981a) considered the basic equation which models groundwater flow and flow of components in a petroleum reservoir, i.e.,

$$\beta \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(\alpha \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\alpha \frac{\partial u}{\partial y}) + q,$$

where u represents the pressure, q is a source/sink term, α is a transmissibility term containing reservoir permeability ($\alpha>0$), β is a storage term containing reservoir porosity ($\beta > 0$), and α and β are considered to be functions of the spatial coordinates. The inverse problem then centres on identification of α and β from observed values of u and q.

Under steady-state conditions, the partial differential equation is reduced to the hyperbolic equation

$$\nabla\alpha\cdot\nabla u + \alpha\Delta u = -q \quad , \quad x \in \Omega\subset\mathbb{R}^n$$

and in a systematic theoretical analysis of this problem, Richter employed a direct approach (i.e., a method involving an approximate solution of the hyperbolic equation). But for estimating α, this has severe practical limitations since it requires knowledge of the derivatives of u.

Richter considered the problem under four different conditions:

(i) $\inf_\Omega |\nabla u| > 0$,

(ii) $\inf_\Omega \Delta u > 0$,

(iii) $\inf_{\Omega}[\max\{|\nabla u|,\Delta u\}] > 0$, and

(iv) $\inf_{\Omega}[\max\{|\nabla u|,|\Delta u|\}] > 0$.

Under each of the first three conditions, he showed that a unique solution α exists for any q , provided that the initial data (prescribed values along a portion of $\partial\Omega$) are appropriately specified, and that α depends continuously on q , ∇u , and Δu. He also described the cases for which the fourth condition will lead to a unique solution.

Examining the first condition, which implies the existence of non-intersecting characteristics, Richter was able to derive a bound which defines the sensitivity of a solution α to perturbations in q and u. He noted, however, that the bound is of little practical value, since the measured pressures upon which the bound condition is based are not known with sufficient accuracy. As well, the bound suggests that an acceptable estimate of α is possible only if the observed u is sufficiently precise to allow accurate approximation of Δu .

The second condition allows for intersecting characteristics, which occur in the neighbourhood of a source or sink (a 'point of degeneracy'). In this case, Richter showed that there is a preferred sense of direction along the characteristics, i.e., depending upon the initial starting point, the solution is drawn along the characteristics toward either the boundary or the point of degeneracy.

Considering the third condition, which allows for the vanishing of the first order derivatives at points where Δu is of the same sign

(thus, u cannot contain both maxima and minima in Ω), Richter discussed the difficulty of prescribing q and boundary conditions on u which will guarantee that the forward solution satisfies $|\nabla u| > 0$ and $\Delta u > 0$ throughout Ω. However, by combining results for the first two conditions, he derived a statement that provided for the possibility of a unique solution. It should be noted that some difficulty was encountered in reproducing the proof of this statement.

Richter then proposed a particularly useful set of test conditions for measuring u, and, under these conditions, showed that a unique solution α exists for the hyperbolic problem without requiring Cauchy data. He also obtained a bound on the stability of the solution in terms of α, q, and relevant properties of Ω.

In a companion paper, Richter (1981b) developed a finite difference method for approximating the solution α numerically, and proved that, under the third condition, the solution does in fact converge.

CHAPTER SEVEN

SUMMARY AND CONCLUSIONS

An intuitive, trial-and-error approach to history-matching can be costly and time-consuming. In an attempt to reduce these factors, considerable efforts have been made to automate the history-matching procedure for implementation on high-speed computers; and several methods based on ex post facto techniques have been developed for this purpose.

This study examined presently available automatic history-matching algorithms reported in journals of petroleum engineering; discussed the viability of each; and explored the fundamental weaknesses of the two most promising methods. In addition, the existence of alternative approaches to solving the inverse problem for analogous, but less complex, situations was noted.

As discussed in Chapter 2, the automatic history-matching algorithms available to petroleum engineers involve

(a) iterative adjustment and regression analysis (Kruger, 1961; Jacquard & Jain, 1965; Jahns, 1966; Dupuy, 1968),

(b) linear and nonlinear programming techniques (Coats et al. 1970; Slater & Durrer, 1971; Thomas et al., 1972; Boberg et al. 1973; Carter et al., 1974),

(c) energy dissipation analysis (Nelson, 1968),

(d) optimal control theory (Chen et al., 1974; Chavent et al., 1973; Wasserman et al., 1975; Watson et al., 1980),

(e) direct solution (Veatch & Thomas, 1971), and

(f) Bayesian estimation (Gavalas et al., 1976).

These publications generally treat only single-phase or pseudo single-phase incompressible reservoir flow, although many would appear to be satisfactory for multiphase flow. A two-phase, incompressible flow problem was investigated by van den Bosch & Seinfeld (1977), but solution methods are suitable only for one-dimensional flow. Veatch & Thomas (1971) developed a direct method for automatic history-matching which is, theoretically, applicable to multiphase, compressible flow. However, for proper consideration of any multiphase flow problem, it is necessary to have knowledge of the relative permeabilities (which are utilized in the reservoir simulator of ex post facto techniques, or, in the case of Veatch & Thomas, directly in the phase equations being solved). While geological samples provide some rough estimate of these values, any history-matching problem should incorporate determination of the relative permeabilities as well as the absolute permeabilities and porosities.

The most serious problem encountered in automatic history-matching is the tendency to construct ill-conditioned systems of equations (i.e., for the problem $A\underline{x} = \underline{b}$, small relative changes in the matrix A or vector $\underline{b}$ produce large relative errors in the solution vector $\underline{x}$). By the very nature of the history-matching problem, inherent uncertainties exist in both A and $\underline{b}$ because they are based on a measured performance history. Further error inevitably arises from the finite word length of the computer. As Rust and Burrus (1972) noted: "the presence of these errors makes it impossible to obtain a meaningful solution by simply applying one of the classical methods to the system by itself. Moreover, if all we know is the system itself, there is no

nonclassical method that will give a meaningful solution" (see also Tikhonov and Arsenin, (1977)). The history-matching problem can therefore be treated only by incorporating some a priori information about the solution vector $\underline{x}$. But due to the gross inaccuracy of any geological estimates (based on scarce physical data), even this requirement cannot be adequately satisfied.

It is obvious, then, that current automatic history-matching methods are unsatisfactory. As expressed by Boberg et al. (1973), as well as by Dr. G.W. Thomas during personal communication, such methods are simply not competitive with manual techniques which an experienced reservoir engineer might employ; and practising reservoir engineers consequently continue history-matching on a trial-and-error basis, with good judgement and intuition providing reasonably reliable results. It would appear, then, that the determination of effective and reliable solutions to the inverse problem requires development of improved methods for estimating the geological structure and properties of the reservoir, and/or adaptation of sophisticated mathematical techniques to allow for a more thorough theoretical treatment of the entire problem.

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APPENDIX

Alternative Derivation of Equations (4.2)

For any i, and a total of N runs, Equation (4.1) yields

$$a_{io}x_o^{(1)} + a_{i1}x_1^{(1)} + a_{i2}x_2^{(1)} + \ldots + a_{iJ}x_J^{(1)} = \varepsilon_i^{(1)}$$

$$a_{io}x_o^{(2)} + a_{i1}x_1^{(2)} + a_{i2}x_2^{(2)} + \ldots + a_{iJ}x_J^{(2)} = \varepsilon_i^{(2)}$$

$$\vdots$$

$$a_{io}x_o^{(N)} + a_{i1}x_1^{(N)} + a_{i2}x_2^{(N)} + \ldots + a_{iJ}x_J^{(N)} = \varepsilon_i^{(N)} .$$

Now, multiply each of the N equations by its respective coefficient of a_{io}, and add the resulting equations. Similarly, multiply each of the N equations by its respective coefficient of a_{i1}, and add these resulting equations. Continuing this process for each coefficient of a_{ij}, $j = 0,1,2,\ldots,J$, yields a total of $J + 1$ simultaneous equations identical to Equation (4.2).